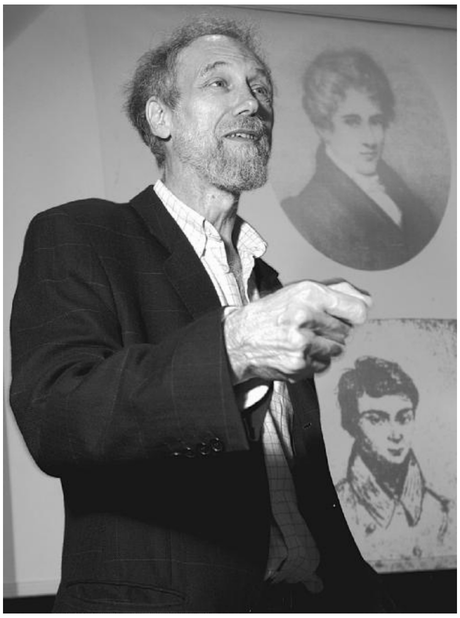
ABEL SYMPOSIA

Edited by the Norwegian Mathematical Society

First row from left: George Elliott, Dorte Olesen, Masamichi Takesaki, Joachim Cuntz, Matilde Marcolli, Alain Connes, Erling Størmer, Ola Bratteli.

Second row from left: Trond Digernes, Ken Dykema, Sergey Neshveyev, Nadia Larsen, Thierry Giordano, Heidi Dahl, Takeshi Katsura, Mikael Rørdam, Søren Eilers, Erik Alfsen, David Evans, Magnus Landstad, Christian Skau, John Rognes.

Last row from left: Anders Hansen, Akitaka Kishimoto, Yosimichi Ueda, Sindre Duedahl, Dimitri Shlyakhtenko, Kjetil Røysland, Stanisław Lech Woronowicz, Georges Skandalis, Toke Meier Carlsen, Eberhard Kirchberg, Erik Bedos, Ragnar Winter, Ryszard Nest.



Alain Connes with predecessors Niels Henrik Abel and Evariste Galois.
Photo credits: Samfoto/Svein Erik Dahl

Ola Bratteli · Sergey Neshveyev
Christian Skau
Editors

Operator Algebras

The Abel Symposium 2004

Proceedings of the First Abel Symposium, Oslo,
September 3–5, 2004

ABEL PRISEN
Springer

Editors

Ola Bratteli
Department of Mathematics
University of Oslo
PB 1053 - Blindern
0316 Oslo
Norway
e-mail: bratteli@math.uio.no

Sergey Neshveyev
Department of Mathematics
University of Oslo
PB 1053 - Blindern
0316 Oslo
Norway
e-mail: sergeyn@math.uio.no

Christian Skau
Department of Mathematical Sciences
Norwegian University of Science and Technology
7491 Trondheim
Norway
e-mail: csk@math.ntnu.no

Library of Congress Control Number: 2006926217

Mathematics Subject Classification (2000): 37A20, 37A60, 46L35, 46L40, 46L54, 46L55, 46L60, 46L80, 46L85, 46L87

ISBN-10 3-540-34196-X Springer Berlin Heidelberg New York
ISBN-13 978-3-540-34196-3 Springer Berlin Heidelberg New York

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable for prosecution under the German Copyright Law.

Springer is a part of Springer Science+Business Media

springer.com

© Springer-Verlag Berlin Heidelberg 2006
Printed in the Netherlands

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Typesetting by the authors and SPi using a Springer LaTeX macro package

Cover design: E. Kirchner, Heidelberg, Germany

Printed on acid-free paper SPIN: 11429630 VA46/3100/SPi 5 4 3 2 1 0

Preface to the Series

The Niels Henrik Abel Memorial Fund was established by the Norwegian government on January 1. 2002. The main objective is to honor the great Norwegian mathematician Niels Henrik Abel by awarding an international prize for outstanding scientific work in the field of mathematics. The prize shall contribute towards raising the status of mathematics in society and stimulate the interest for science among school children and students. In keeping with this objective the board of the Abel fund has decided to finance one or two Abel Symposia each year. The topic may be selected broadly in the area of pure and applied mathematics. The Symposia should be at the highest international level, and serve to build bridges between the national and international research communities. The Norwegian Mathematical Society is responsible for the events. It has also been decided that the contributions from these Symposia should be presented in a series of proceedings, and Springer Verlag has enthusiastically agreed to publish the series. The board of the Niels Henrik Abel Memorial Fund is confident that the series will be a valuable contribution to the mathematical literature.

Ragnar Winther
Chairman of the board of the Niels Henrik Abel Memorial Fund

Preface

The theme of this symposium was operator algebras in a wide sense. In the last 40 years operator algebras has developed from a rather special discipline within functional analysis to become a central field in mathematics often described as "non-commutative geometry" (see for example the book "Non-Commutative Geometry" by the Fields medalist Alain Connes). It has branched out in several sub-disciplines and made contact with other subjects like for example mathematical physics, algebraic topology, geometry, dynamical systems, knot theory, ergodic theory, wavelets, representations of groups and quantum groups. Norway has a relatively strong group of researchers in the subject, which contributed to the award of the first symposium in the series of Abel Symposia to this group. The contributions to this volume give a state-of-the-art account of some of these sub-disciplines and the variety of topics reflect to some extent how the subject has branched out. We are happy that some of the top researchers in the field were willing to contribute.

The basic field of operator algebras is classified within mathematics as part of functional analysis. Functional analysis treats analysis on infinite dimensional spaces by using topological concepts. A linear map between two such spaces is called an operator. Examples are differential and integral operators. An important feature is that the composition of two operators is a non-commutative operation. It is often convenient not just to consider a single operator, but a whole class of operators which form an algebra and satisfy some technical conditions. The basic theory of Operator algebras encompasses C^*-algebras and von Neumann algebras. The study of C^*-algebras could be called non-commutative topology and the study of von Neumann algebras non-commutative measure theory, since this study reduces to the study of topology and measure theory in the case that the algebras are abelian.

The symposium took place in Oslo, September 3–5, 2004, and was organized by

- Ola Bratteli, University of Oslo
- Alain Connes, Collège de France, Paris

- Joachim Cuntz, Westfälische Wilhelms-Universität Münster
- Sergei Neshveyev, University of Oslo
- Christian Skau, Norwegian University of Science and Technology, Trondheim
- Erling Størmer, University of Oslo

The symposium was dedicated to the memory of Gert Kjærgaard Pedersen, the pater familias of the operator algebraists in Denmark, who was invited to give a talk, but died March 15, 2004. One of his last contributions to mathematics is published in these proceedings.
The following senior researchers from abroad participated and all gave talks:

- Alain Connes, Paris
- Joachim Cuntz, Münster
- Ken Dykema, Texas A&M
- Søren Eilers, Copenhagen
- George Elliott, Toronto
- David Evans, Cardiff
- Thierry Giordano, Ottawa
- Takeshi Katsura, Sapporo
- Eberhard Kirchberg, Berlin
- Akitaka Kishimoto, Sapporo
- Matilde Marcolli, Bonn
- Ryszard Nest, Copenhagen
- Dorte Olesen, Copenhagen
- Mikael Rørdam, Odense
- Dimitri Shlyakhtenko, UCLA
- Georges Skandalis, Paris
- Masamichi Takesaki, UCLA
- Yoshimichi Ueda, Kyushu
- Stanisław Lech Woronowicz, Warsaw

Senior researchers and postdocs from Oslo and Trondheim who participated:

- Erik Alfsen
- Erik Bedos
- Ola Bratteli
- Toke Meier Carlsen
- Trond Digernes
- Magnus Landstad
- Nadia Larsen
- Sergey Neshveyev
- Christian Skau
- Erling Størmer

Doctoral students from Oslo and Trondheim who participated:

- Sindre Duedahl
- Kjetil Røysland
- Heidi Dahl

More information about the symposium may be found at this web page:

http://abelsymposium.no/2004

Oslo and Trondheim 27 March 2006

Ola Bratteli   Sergei Neshveyev   Christian Skau

Contents

Interpolation by Projections in C^*-Algebras

Lawrence G. Brown and Gert K. Pedersen*

Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA,
`lgb@math.purdue.edu`

Dedicated to the memory of Gert. K. Pedersen

Note from author L.G.B.: This paper was begun in 2002 and was mainly completed in that year. There were some possible small changes still under discussion. In this version I have made only very minor changes that I'm sure Gert would have approved of.

Summary. If x is a self-adjoint element in a unital C^*-algebra $\mathcal{A}$, and if p_δ and q_δ denote the spectral projections of x corresponding to the intervals $]\delta, \infty[$ and $]-\infty, -\delta[$, we show that there is a projection p in $\mathcal{A}$ such that $p_\delta \le p \le \mathbf{1} - q_\delta$, provided that $\delta > \operatorname{dist}\{x, A_{\mathrm{sa}}^{-1}\}$. This result extends to unbounded operators affiliated with a C^*−algebra, and has applications to certain other distance functions.

1 Introduction

1.1

Let x be an operator on a Hilbert space $\mathcal{H}$ with polar decomposition $x = v|x|$, and for each $\delta \ge 0$ let e_δ and f_δ denote the spectral projections of $|x|$ and $|x^*|$, respectively, corresponding to the interval $]\delta, \infty[$. Practically the first observation to be made in single operator theory is that e_δ and f_δ are Murray–von Neumann equivalent; in fact, $ve_\delta v^* = f_\delta$. The second observation is that $\mathbf{1} - e_\delta$ and $\mathbf{1} - f_\delta$ need not be equivalent if $\mathcal{H}$ is infinite dimensional; in fact, $(\mathbf{1} - e_0)\mathcal{H} = \ker x$ and $(\mathbf{1} - f_0)\mathcal{H} = \ker x^*$, and these spaces may have widely different dimensions. If, however, $\mathbf{1} - e_\delta = w^*w$ and $\mathbf{1} - f_\delta = ww^*$ for some partial isometry w, then $u = w + ve_\delta$ is a unitary conjugating e_δ to f_δ. Equivalently phrased, the operator xe_δ can now be written $xe_\delta = u|xe_\delta|$ with a unitary u.

*Supported in part by SNF, Denmark.

If x belongs to an algebra $\mathcal{A}$ of operators on $\mathcal{H}$ the questions above can all be reformulated, asking now whether the unitary u can be chosen in $\mathcal{A}$. In the case of a von Neumann algebra $\mathcal{A}$ this question was solved by C.L. Olsen in [14], using the distance to the set $\mathcal{A}^{-1}$ of invertible elements,

$$\alpha(x) = \operatorname{dist}\{x, \mathcal{A}^{-1}\}.$$

The answer is that $xe_\delta = u|xe_\delta|$ for some unitary u in $\mathcal{A}$ when $\delta > \alpha(x)$.

If $\mathcal{A}$ is only a C^*-algebra (always assumed unital in this paper unless otherwise specified) some care must be taken to formulate the question, because the spectral projections of an element do not (necessarily) belong to the algebra. However, if $x = v|x|$ is the polar decomposition then the element $x_f = vf(|x|) \in \mathcal{A}$ for every continuous function f vanishing at zero. We can therefore ask whether $x_f = u|x_f|$ for some unitary u in $\mathcal{A}$, provided that f vanishes on some interval $[0, \delta]$. In fact, this is equivalent to the demand that $ue_\delta = ve_\delta$ (whence also $f_\delta u = f_\delta v$), so that the partial isometry ve_δ has a unitary extension u in A. Combining a couple of highly technical lemmas this problem was solved in [19, Theorem 2.2] and [15, Theorem 5] with the same answer as in the von Neumann algebra case: If $\delta > \alpha(x)$ then for any continuous function f vanishing on $[0, \delta]$ we have $x_f = u|x_f|$ for some unitary u in $\mathcal{A}$. If $\delta < \alpha(x)$ no extension is possible.

The limit case $\delta = \alpha(x)$ is left undecided: Sometimes a unitary extension exists, sometimes not. For von Neumann algebras the index of x is a natural obstruction, but in general the situation is more subtle. Closer investigation shows that (outside finite AW^*-algebras) it is very unlikely that every x in the closure of the invertible elements in some C^*-algebra can be written in the form $x = u|x|$ with a unitary u in $\mathcal{A}$, cf. [8] and [16].

1.2

If $\mathcal{A}_l^{-1}$ denotes the set of left invertible elements in a C^*-algebra $\mathcal{A}$ we can define the function

$$\alpha_l(x) = \operatorname{dist}\{x, \mathcal{A}_l^{-1}\}.$$

It was shown in [17, Theorem 7.1] that if $\delta > \alpha_l(x)$ then any element $x_f = vf(|x|)$ can be written as $x_f = u|x_f|$ for some isometry u in $\mathcal{A}$, provided that f vanishes on $[0, \delta]$. The proof, however, is not very illuminating, since it quickly reduces to the regular case. Evidently there is also a symmetric result for the set $\mathcal{A}_r^{-1}$ of right invertible elements and co-isometries in $\mathcal{A}$, using the function $x \to \alpha_l(x^*)$.

A much more serious approach was needed to handle the set $\mathcal{A}_q^{-1}$ of quasi-invertible elements. Recall from [4] that $a \in \mathcal{A}_q^{-1}$ if $(\mathbf{1} - ba)\mathcal{A}(\mathbf{1} - ab) = 0$ for some b in $\mathcal{A}$. If we can choose $b = a^*$ then a is an extreme point in the unit ball of $\mathcal{A}$ and may be regarded as a partial isometry which is "maximally extended". A general quasi-invertible element always has the form $a = xuy$

with x, y in $\mathcal{A}^{-1}$ and u an extreme partial isometry, cf. [4, Theorem 1.1]. Now define

$$\alpha_q(x) = \text{dist}\,\{x, \mathcal{A}_q^{-1}\}.$$

By [4, Theorem 2.2] we can then find an extreme partial isometry u in $\mathcal{A}$ such that $x_f = u|x_f|$, whenever $x_f = vf(|x|)$ and f is a continuous function vanishing on an interval $[0, \delta]$ with $\delta > \alpha_q(x)$. Equivalently, $ue_\delta = ve_\delta$ and $f_\delta u = f_\delta v$ if $\delta > \alpha_q(x)$.

1.3

Corresponding to the three distance functions mentioned above we have three classes of C^*-algebras, characterized by the norm density of the three subsets $\mathcal{A}^{-1}$, $\mathcal{A}_l^{-1} \cup \mathcal{A}_r^{-1}$ and $\mathcal{A}_q^{-1}$. These are known, respectively, as C^*-algebras *of stable rank one*, *isometrically rich* C^*-algebras and *extremally rich* C^*-algebras. In such an algebra the polar decomposition of any element $x_f = vf(|x|)$ can be "upgraded", i.e. v can be replaced by a unitary, an isometry or a co-isometry, or an extreme partial isometry, if only f vanishes in some (small) neighbourhood of zero.

In [3] we introduced the class of C^*-algebras of *real rank zero* as those C^*-algebras $\mathcal{A}$ for which the set $\mathcal{A}_{\text{sa}}^{-1}$ of invertible self-adjoint elements in the algebra was dense in $\mathcal{A}_{\text{sa}}$. (As for the other classes, a non-unital C^*-algebra has real rank zero if the unitized algebra fulfills the criterion.) Over the years a considerable theory has been developed for these classes of C^*-algebras, the real rank zero being the most "AF-like," the stable rank one algebras the most "finite."

One of the surprising phenomena (and the guiding principle in [6] and [7]) has been the patent, albeit subtle, similarity between C^*-algebras of stable rank one and C^*-algebras of real rank zero. For example, a theorem in K-theory that is valid for one class stands a very good chance also of being valid for the other class, but with a change of degree from $K_n(\mathcal{A})$ to $K_{n+1}(\mathcal{A})$. Related to this is the extension theory for the two classes. In both cases there is a known obstruction for an extension to be in the same class as the ideal and the quotient. For stable rank one algebras it is the lifting of unitaries from the quotient, for real rank zero the lifting of projections (equivalently, the lifting of self-adjoint unitaries).

The distance function

$$\alpha_r(x) = \text{dist}\,\{x, \mathcal{A}_{\text{sa}}^{-1}\}$$

provides another parallel case. Thus we show in [6, Theorem 2.3] for a general (unital) C^*-algebra $\mathcal{A}$ that the self-adjoint part of the largest ideal $I_{\text{RR0}}(\mathcal{A})$ of $\mathcal{A}$ of real rank zero consists precisely of elements x in $\mathcal{A}_{\text{sa}}$ such that $\alpha_r(x + y) = \alpha_r(y)$ for every y in $\mathcal{A}_{\text{sa}}$. This should be compared to Rørdam's characterization in [19, Propositions 4.1 & 4.2] of the largest ideal $I_{\text{sr1}}(\mathcal{A})$ of stable rank one in $\mathcal{A}$, as consisting precisely of those elements x in $\mathcal{A}$ such that $\alpha(x + y) = \alpha(y)$ for every y in $\mathcal{A}$.

1.4

The main result in this paper, Theorem 3, is the exact analogue of the three polar decomposition results mentioned above, but now for self-adjoint elements only. This may at first seem odd, because $x = x^*$ implies that $e_\delta = f_\delta$ for all δ, and if $x = v|x|$ then $v = v^*$, so that $v = p - q$ for a pair of orthogonal projections. But if $\mathcal{A}$ is only a C^*-algebra it is still meaningful and interesting to ask whether v can be extended to a self-adjoint unitary, i.e. a symmetry in the algebra. Evidently this is so if zero is an isolated point in $\mathrm{sp}(x)$, but there are many other cases. For von Neumann algebras there is no problem; but then von Neumann algebras all have real rank zero. For C^*-algebras not of real rank zero there may not be very many projections around, hence also not very many symmetries. Our result may serve to locate these projections and control their behaviour.

Our result can also be interpreted as an interpolation, and we shall most often phrase it as such: If p_δ and q_δ denote the spectral projections of x corresponding to the intervals $]\delta, \infty[$ and $]-\infty, -\delta[$ (so that $p_\delta + q_\delta = e_\delta$ in the previous terminology), we show that there is a projection p in the algebra such that

$$p_\delta \le p \le \mathbf{1} - q_\delta,$$

provided that $\delta > \alpha_r(x)$. For C^*-algebras of real rank zero, where $\alpha_r(x) = 0$ for every x, this result was obtained in [2]. In fact it was proved in [2, Theorem 1] that $\mathcal{A}$ has real rank zero (in the sense that it satisfies one of the equivalent conditions HP or FS from [3]) if and only if it has *interpolation of projections*, IP, in the sense that whenever $\overline{p}$ is a compact and p° an open projection in $\mathcal{A}^{**}$ with $\overline{p} \le p^\circ$, then $\overline{p} \le p \le p^\circ$ for some projection p in $\mathcal{A}$.

2 Main Results

Lemma 1. *Let x be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and for $\delta > 0$ define the continuous functions*

$$c_\delta(t) = t, \qquad d_\delta(t) = (\delta^2 - t^2)^{1/2} \qquad \textit{for} \quad |t| \le \delta, \tag{1}$$
$$c_\delta(t) = \delta \,\mathrm{sign}\, t, \qquad d_\delta(t) = 0 \qquad \textit{for} \quad |t| \ge \delta. \tag{2}$$

Then $\mathrm{sp}(a) \cap]-\delta, \delta[= \emptyset$, where a is the operator matrix

$$a = \begin{pmatrix} c_\delta(x) & d_\delta(x) \\ d_\delta(x) & -x \end{pmatrix}.$$

Proof. If $\lambda \in \mathrm{sp}(a)$ then for some t in $\mathrm{sp}(x)$ we have

$$(c_\delta(t) - \lambda)(-t - \lambda) - d_\delta(t)^2 = 0.$$

If $\delta \le |t|$ this equation simply becomes $(\delta \operatorname{sign} t - \lambda)(t + \lambda) = 0$, with the solutions $\lambda = \delta \operatorname{sign} t$ and $\lambda = -t$. It follows that $|\lambda| \ge \delta$.

If $|t| \le \delta$ we obtain the equation $(t - \lambda)(-t - \lambda) - (\delta^2 - t^2) = 0$, or $\lambda^2 - \delta^2 = 0$, with the solutions $\lambda = \pm\delta$; so that again $|\lambda| \ge \delta$. □

Definition 2. *As usual, cf. [4, Section 1], given a self-adjoint element x in a C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ we define the constant*

$$\begin{aligned} m(x) &= \sup\{\varepsilon \ge 0 \mid]-\varepsilon, \varepsilon[\cap \operatorname{sp}(x) = \emptyset\} \\ &= \inf\{\|x\xi\| \mid \xi \in \mathcal{H} : \|\xi\| = 1\} \\ &= \operatorname{dist}\{x, \left(\mathcal{A}_{\mathrm{sa}} \setminus \mathcal{A}_{\mathrm{sa}}^{-1}\right)\}. \end{aligned}$$

Note that $m(x) = m(|x|)$, so that for a general (non self-adjoint) element x in $\mathcal{A}$ we can define $m(x) = m(|x|)$. Alternatively, we can use the second expression, which makes sense for all operators. It is an easy consequence of the open mapping theorem that x is invertible if and only if $m(x) > 0$ and $m(x^*) > 0$ [since then $\ker x = 0$ and $x(\mathcal{H}) = \mathcal{H}$].

Theorem 3. *Let x be a self-adjoint element in a unital C^*–algebra $\mathcal{A}$, and for $\delta \ge 0$ denote by p_δ and q_δ the spectral projections of x (in $\mathcal{A}^{**}$) corresponding to the intervals $]\delta, \infty[$ and $]-\infty, -\delta[$, respectively. If $\delta > \alpha_r(x)$ there is a projection p in $\mathcal{A}$ such that*

$$p_\delta \le p \le \mathbf{1} - q_\delta.$$

Equivalently, for any continuous function f vanishing on the interval $[-\delta, \delta]$ and such that $f(t) \operatorname{sign} t \ge 0$ for all t we have $f(x) = (2p - \mathbf{1})|f(x)|$ in $\mathcal{A}$.

If $\delta < \alpha_r(x)$ there are no projections p in $\mathcal{A}$ such that $p_\delta \le p \le \mathbf{1} - q_\delta$, and no symmetries u in $\mathcal{A}$ such that $f(x) = u|f(x)|$ if we choose $f(t) = \operatorname{sign} t\ (|t| - \delta)_+$.

Proof. By assumption we can find y in $\mathcal{A}_{\mathrm{sa}}^{-1}$ with $\|x - y\| < \delta$. With c_δ and d_δ as in Lemma 1 this means that the operator matrix

$$b = \begin{pmatrix} c_\delta(x) & d_\delta(x) \\ d_\delta(x) & -y \end{pmatrix}$$

is still invertible (in $\mathbb{M}_2(\mathcal{A})$), because $m(b) \ge \delta - \|x - y\| > 0$, cf. Definition 2. Consequently also the matrix

$$\begin{pmatrix} \mathbf{1} & d_\delta(x)y^{-1} \\ 0 & \mathbf{1} \end{pmatrix} b \begin{pmatrix} \mathbf{1} & 0 \\ y^{-1}d_\delta(x) & \mathbf{1} \end{pmatrix} = \begin{pmatrix} c_\delta(x) + d_\delta(x)y^{-1}d_\delta(x) & 0 \\ 0 & -y \end{pmatrix}$$

is invertible in $\mathbb{M}_2(\mathcal{A})$, from which we conclude that the self-adjoint element

$$z = c_\delta(x) + d_\delta(x)y^{-1}d_\delta(x)$$

is invertible in $\mathcal{A}$.

By construction we have

$$p_\delta d_\delta(x) = q_\delta d_\delta(x) = 0, \quad p_\delta c_\delta(x) = \delta p_\delta, \quad q_\delta c_\delta(x) = -\delta q_\delta.$$

Therefore $p_\delta z = \delta p_\delta$ and $q_\delta z = -\delta q_\delta$. If p denotes the spectral projection of z corresponding to the interval $]0,\infty[$, then $p \in \mathcal{A}$ since $0 \notin \mathrm{sp}(z)$. From the equations above we see that $p_\delta p = p_\delta$ and $q_\delta p = 0$, whence $p_\delta \le p \le \mathbf{1} - q_\delta$.

If f is a continuous function vanishing on $[-\delta,\delta]$ such that $f(t)\,\mathrm{sign}\,t \ge 0$ for all t, then by spectral theory

$$\begin{aligned} f(x) &= (p_\delta + q_\delta)f(x) = (p_\delta - q_\delta)|f(x)| \\ &= (2p-\mathbf{1})(p_\delta + q_\delta)|f(x)| = (2p-\mathbf{1})|f(x)|. \end{aligned}$$

Assume now that for some δ we have a projection p in $\mathcal{A}$ such that $p_\delta \le p \le \mathbf{1} - q_\delta$. Put $u = 2p - \mathbf{1}$. Then with $f(t) = \mathrm{sign}\,t\,(|t|-\delta)_+$ and $\varepsilon > 0$ consider the element $y = u\,((|x|-\delta)_+ + \varepsilon\mathbf{1})$. Evidently $y \in \mathcal{A}_{\mathrm{sa}}^{-1}$ and $\|f(x) - y\| \le \varepsilon$. Consequently

$$\|x - y\| \le \|x - f(x)\| + \varepsilon \le \delta + \varepsilon.$$

Since ε is arbitrary we conclude that $\alpha_r(x) \le \delta$. This proves the last statement in the theorem. □

Corollary 4. *For every self-adjoint element x in a unital C^*-algebra $\mathcal{A}$ put $\alpha = \alpha_r(x)$ and define $x_\alpha = c_\alpha(x)$, where $c_\alpha(t) = \mathrm{sign}\,t\,(|t| \wedge \alpha)$ as in Lemma 1. Then $x - x_\alpha \in (\mathcal{A}_{\mathrm{sa}}^{-1})^=$, $\|x - x_\alpha\| = \|x\| - \alpha$ and $\|x_\alpha\| = \alpha = \alpha_r(x_\alpha)$.*

Example 5. It is easy to find examples where no projections exist in the limit $\delta = \alpha_r(x)$. If $\Omega = [-1,1] \cup \{1 + 1/n \mid n \in \mathbb{N}\} \cup \{-1 - 1/n \mid n \in \mathbb{N}\}$ and $\mathcal{A} = C(\Omega)$, then with $x = \mathrm{id}$ we obtain a self-adjoint element with $\alpha_r(x) = 1$ (but $\|x\| = 2$). The spectral projections p_1 and q_1 correspond to the characteristic functions for the sets $\{1 + 1/n \mid n \in \mathbb{N}\}$ and $\{-1 - 1/n \mid n \in \mathbb{N}\}$, respectively, so there is no projection p in $\mathcal{A}$ such that $p_1 \le p \le \mathbf{1} - q_1$.

Definition 6 (Unbounded Operators). *Let x be an unbounded self-adjoint operator in a Hilbert space $\mathcal{H}$. We say that x is affiliated with a non-unital C^*-algebra $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ if $(x - \lambda\mathbf{1})^{-1} \in \mathcal{A}$ for every λ outside* $\mathrm{sp}(x)$. *Equivalently, $(x - it\mathbf{1})^{-1} \in \mathcal{A}$ whenever $t \ne 0$. It follows that $f(x) \in \mathcal{A}$ for every f in $C_0(\mathbb{R})$. In addition we demand that the subalgebra $\{f(x) \mid f \in C_0(\mathbb{R})\}$ contain an approximate unit for $\mathcal{A}$, which is equivalent to the demand that $(\mathbf{1} + x^2)^{-1}$ be a strictly positive element in $\mathcal{A}$. From this extra condition we conclude that the multiplier algebra $M(\mathcal{A})$ of $\mathcal{A}$ contains every element of the form $f(x)$ where $f \in C_b(\mathbb{R})$.*

The set $\mathcal{A}^{\text{aff}}$ of affiliated operators is not an algebra (in general), not even a vector space, but $x - a \in \mathcal{A}^{\text{aff}}$ for every x in $\mathcal{A}^{\text{aff}}$ and a in $M(\mathcal{A})_{\text{sa}}$. To see this we take $|t| > \|a\|$. Then

$$(x - a - it\mathbf{1})^{-1} = \big((x - it\mathbf{1})(\mathbf{1} - (x - it\mathbf{1})^{-1}a)\big)^{-1}$$
$$= \sum_{n=0}^{\infty} \big((x - it\mathbf{1})^{-1}a\big)^{n} (x - it\mathbf{1})^{-1} \in \mathcal{A}.$$

On the other hand, if $(x - a - it\mathbf{1})^{-1} \in \mathcal{A}$ for some t then also

$$(x - a - is\mathbf{1})^{-1} = \big((x - a - it\mathbf{1})(\mathbf{1} - (x - a - it\mathbf{1})^{-1})i(s - t)\big)^{-1} \in \mathcal{A}$$

for $|s - t| < |t|$, since $\|(x - a - it\mathbf{1})^{-1}\| \leq |t|^{-1}$. Taken together this means that $(x - a - it\mathbf{1})^{-1} \in \mathcal{A}$ for all $t \neq 0$, whence $x - a \in \mathcal{A}^{\text{aff}}$.

For each x affiliated with $\mathcal{A}$ and every $\gamma > 0$ we define the cut-down operator $x_\gamma = c_\gamma(x)$ in $M(\mathcal{A})$, where $c_\gamma(t) = \operatorname{sign} t\ (|t| \wedge \gamma)$ as in Lemma 1 and Corollary 4. On these operators we apply the function $\alpha_r(\cdot)$ relative to the unital C^*-algebra $M(\mathcal{A})$.

Lemma 7. *If $\alpha_r(x_\gamma) < \gamma$ for some $\gamma > 0$ then $\alpha_r(x_\beta) = \alpha_r(x_\gamma)$ for all $\beta > \gamma$.*

Proof. By assumption we can find δ such that $\alpha_r(x_\gamma) < \delta < \gamma$, and then consider the spectral projections p_δ and q_δ of x. However, p_δ and q_δ can also be regarded as spectral projections of x_γ and of x_β, still corresponding to the intervals $]\delta, \infty[$ and $]-\infty, -\delta[$.

It is therefore easy to deduce the result from Theorem 3. □

Definition 8. *If x is a self-adjoint operator affiliated with a non-unital C^*-algebra $\mathcal{A}$ we define $\alpha_r(x)$ to be the infimum of numbers $\|a\|$, where $a \in M(\mathcal{A})_{\text{sa}}$ such that $x - a$ is invertible (whence $(x - a)^{-1} \in \mathcal{A}$). If no such a exists we set $\alpha_r(x) = \infty$. Loosely speaking we may refer to $\alpha_r(x)$ as the distance between x and the invertible operators affiliated with $\mathcal{A}$. At least we see that $\alpha_r(x + b) \leq \alpha_r(x) + ||b||$ for every x in $\mathcal{A}^{\text{aff}}$ and b in $M(\mathcal{A})_{\text{sa}}$.*

Theorem 9. *Let x be a self-adjoint operator affiliated with a non-unital C^*-algebra $\mathcal{A}$. For every $\delta > \alpha_r(x)$ there is then a projection p in $M(\mathcal{A})$ interpolating the spectral projections p_δ and $\mathbf{1} - q_\delta$ of x. Moreover,*

$$\alpha_r(x) = \inf\{\gamma > 0 \mid \alpha_r(x_\gamma) < \gamma\}.$$

Proof. If $\alpha_r(x) < \delta < \infty$ we can find a in $M(\mathcal{A})_{\text{sa}}$ with $\|a\| < \delta$ such that $y = x - a$ is invertible (and $y^{-1} \in \mathcal{A}$). With c_δ and d_δ as in Lemma 1 we obtain an operator matrix, where now the $(2,2)$-corner is unbounded; but it is still true that the spectrum of the matrix misses the open interval $]-\delta, \delta[$. This means that when in the proof of Theorem 3 we define the operator matrix b, where the $(2,2)$- corner is $-y$, we have an unbounded, but invertible operator.

The element $z = c_\delta(x) + d_\delta(x)y^{-1}d_\delta(x)$ is therefore again invertible, but also bounded, and $z \in M(\mathcal{A})$. The rest of the proof proceeds as before to prove that there is a projection p in $M(\mathcal{A})$ interpolating the spectral projections p_δ and $\mathbf{1} - q_\delta$ of x.

Since the projection p found above also interpolates the spectral projections of the cut-down elements x_γ when $\gamma > \delta$, it follows that $\alpha_r(x_\gamma) \le \delta < \gamma$. As δ can be chosen arbitrarily close to $\alpha_r(x)$ this implies that $\inf\{\gamma > 0 \mid \alpha_r(x_\gamma) < \gamma\} \le \alpha_r(x)$.

To prove the reverse inequality we note that if for some γ we have $\alpha_r(x_\gamma) < \beta < \gamma$ for some β, then $p_\beta \le p \le \mathbf{1} - q_\beta$ by Theorem 3, with p a projection in $M(\mathcal{A})$. Consequently $x - x_\beta = f(x) = (2p - \mathbf{1})|f(x)|$, where $f(t) = \operatorname{sign} t\ (|t| - \beta)_+$, and the self-adjoint element $(2p - \mathbf{1})(|f(x)| + \varepsilon\mathbf{1})$ is invertible for every $\varepsilon > 0$ and affiliated with $\mathcal{A}$. Since $\|x_\beta\| \le \beta$ it follows that $\alpha_r(x) \le \beta + \varepsilon$, whence in the limit $\alpha_r(x) < \gamma$. □

Remarks

The problems encountered when trying to remove zero from the spectrum of certain differential (Dirac) operators are well documented in the literature, see e.g. [9, 10, 11, 12, 13]. We hope that our result can be of some use in this context.

Clearly we do not need the whole multiplier algebra to formulate the results in Theorem 9. What is required is a unital C^*-algebra $\mathcal{B}$ such that $f(x) \in \mathcal{B}$ for each f in $C(\mathbb{R} \cup \pm\infty)$ (the two-point compactification of $\mathbb{R}$), but also such that $f(x + b) \in \mathcal{B}$ if $b \in \mathcal{B}_{\text{sa}}$.

On the other hand we see from the proof of Theorem 9 that if $y = x - a$ is an invertible bounded perturbation of x (so that $a \in M(\mathcal{A})_{\text{sa}}$) then the invertible element $z = x_\delta + d_\delta(x)y^{-1}d_\delta(x)$ is an $\mathcal{A}$-perturbation of x_δ, so that we can assert that x_δ is invertible in the corona algebra $M(\mathcal{A})/\mathcal{A}$.

3 Some Applications

3.1 Distance to the Symmetries

The formula for the distance between an element x in a C^*-algebra $\mathcal{A}$ and the group $\mathcal{U}(\mathcal{A})$ of unitaries was proved in [19, Theorem 2.7], see also [15, Theorem 10], in complete analogy with the formula for the von Neumann algebra case found by C.L. Olsen in [14]. The same formula, but with $\alpha_q(\cdot)$ replacing $\alpha(\cdot)$, describes the distance to the set $\mathcal{E}(\mathcal{A})$ of extreme partial isometries in $\mathcal{A}$ by [5, Theorem 3.1]. We show below that the exact same formula – but now with $\alpha_r(\cdot)$ replacing $\alpha(\cdot)$ – describes the distance between a self-adjoint element x and the set $\mathcal{S}(\mathcal{A})$ of symmetries in $\mathcal{A}$. As for unitaries in C^*-algebras one can not in general hope to find an approximant to x in $\mathcal{S}(\mathcal{A})$, Example 5 provides a case in point, but in special cases they exist, cf. Corollary 11 and Proposition 12.

Proposition 10. *Let x be a non-invertible self-adjoint element in a unital C^*-algebra $\mathcal{A}$ and let $\mathcal{S}(\mathcal{A})$ denote the set of symmetries in $\mathcal{A}$. Then $[-\alpha_r(x), \alpha_r(x)] \subset \mathrm{sp}(ux)$ for every u in $\mathcal{S}(\mathcal{A})$. Moreover,*

$$\mathrm{dist}\,\{x, \mathcal{S}(\mathcal{A})\} = \max\,\{\|x\| - 1,\ \alpha_r(x) + 1\}.$$

Proof. If $u \in \mathcal{S}(\mathcal{A})$ and $\lambda \notin \mathrm{sp}(ux)$ for some real number λ, then $\lambda \neq 0$ and $\lambda\mathbf{1} - ux \in \mathcal{A}^{-1}$, whence $\lambda u - x \in \mathcal{A}_{\mathrm{sa}}^{-1}$. Therefore $|\lambda| = \|\lambda u\| \geq \alpha_r(x)$, proving the first statement.

We see from above that

$$\|x - u\| = \|\mathbf{1} - ux\| \geq \rho(\mathbf{1} - ux) \geq 1 + \alpha_r(x);$$

and evidently $\|x - u\| \geq \|x\| - 1$, so that we have the inequality

$$\mathrm{dist}\,\{x, \mathcal{S}(\mathcal{A})\} \geq \max\,\{\|x\| - 1,\ \alpha_r(x) + 1\}.$$

To prove the reverse inequality we take $\delta > \alpha_r(x)$ and find a projection p in $\mathcal{A}$ with $p_\delta \leq p \leq \mathbf{1} - q_\delta$ using Theorem 3. Then with $u = 2p - \mathbf{1}$ we have

$$\begin{aligned}\|x - u\| &= \|(x - u)(p_\delta + q_\delta + (\mathbf{1} - p_\delta - q_\delta))\| \\ &= \max\,\{\|(x - u)p_\delta\|,\ \|(x - u)q_\delta\|,\ \|(x - u)(\mathbf{1} - p_\delta - q_\delta)\|\} \\ &\leq \max\,\{\|x_+\| - 1,\ 1 - \delta,\ \|x_-\| - 1,\ 1 + \delta\}.\end{aligned}$$

Since δ can be chosen arbitrarily near $\alpha_r(x)$ the result follows. □

Corollary 11. *If $\alpha_r(x) < \|x\| - 2$ there is a symmetry u in $\mathcal{A}$ such that*

$$\|x - u\| = \mathrm{dist}\,\{x, \mathcal{S}(A)\} = \|x\| - 1$$

Proof. By assumption we can find δ such that $\alpha_r(x) < \delta < \|x\| - 2$. Choosing the symmetry u as in Proposition 10 this means that $\|x - u\| = \|x\| - 1$, as desired. □

Proposition 12. *Let x be self-adjoint and invertible in a unital C^*-algebra $\mathcal{A}$ with polar decomposition $x = u|x|$. Then with $m(x)$ as in Definition 2 we have*

$$\mathrm{dist}\,\{x,\ \mathcal{S}(\mathcal{A})\} = \max\,\{\|x\| - 1,\ 1 - m(x)\} = \|x - u\|.$$

Proof. If $w \in \mathcal{S}(\mathcal{A})$ then evidently $\|x - w\| \geq \|x\| - 1$. Moreover, for each unit vector ξ we have $\|x - w\| \geq \|w(\xi)\| - \|x(\xi)\| = 1 - \|x(\xi)\|$, proving the inequality

$$\mathrm{dist}\,\{x,\ \mathcal{S}(\mathcal{A})\} \geq \max\,\{\|x\| - 1,\ 1 - m(x)\}.$$

On the other hand, with $x = u|x|$ we have by spectral theory that

$$\|x - u\| = \sup\,\{|t - \mathrm{sign}\, t| \mid t \in \mathrm{sp}(x)\} = \max\,\{\|x\| - 1,\ 1 - m(x)\},$$

as desired. □

3.2 The λ-Function

For every element x in the unit ball $\mathcal{A}^1$ of a unital C^*-algebra $\mathcal{A}$ the number $\lambda(x)$ is defined as the supremum of all λ in $[0,1]$ such that $x = \lambda u + (1-\lambda)y$ for some extreme point u in $\mathcal{A}^1$ and some arbitrary y in $\mathcal{A}^1$. This *λ-function* on $\mathcal{A}^1$ was completely determined in [5, Theorem 3.7] in terms of the numbers $\alpha_q(x)$ and $m_q(x)$. In particular it was shown that $\mathcal{A}$ has the *λ-property* ($\lambda(x) > 0$ for every x in $\mathcal{A}^1$) if and only if $\lambda(x) \geq 1/2$ for every x, which happens precisely when $\mathcal{A}$ is extremally rich, cf. Section 1.3.

The simpler cases of unitaries or isometries were solved earlier in [17, Theorems 5.1, 5.4 & 8.1] with formulae resembling the case above. The relevant function here is the *unitary λ-function*, $\lambda_u(x)$, defined on A^1 as the supremum of all λ in $[0,1]$ such that $x = \lambda u + (1-\lambda)y$ for some unitary u in A and y in A^1. We found that $\lambda(x) > 0$ for every x in A^1 precisely when A has stable rank one, in which case actually $\lambda_u(x) \geq 1/2$.

We now define the *real λ-function* $\lambda_r(x)$ on $\mathcal{A}^1_{\rm sa}$ to be the supremum of all λ in $[0,1]$ such that $x = \lambda u + (1-\lambda)y$ for some symmetry u in $\mathcal{S}(\mathcal{A})$ and y in $\mathcal{A}^1_{\rm sa}$. As we shall see, the form of this function is completely analogous to the classical λ-function, with $\alpha_r(\cdot)$ and $m(\cdot)$ replacing $\alpha_q(\cdot)$ and $m_q(\cdot)$; and a C^*-algebra $\mathcal{A}$ has the *real λ-property* ($\lambda_r(x) > 0$ for every x in $\mathcal{A}^1_{\rm sa}$) if and only if $\lambda_r(x) \geq 1/2$ for every x, which happens precisely when $\mathcal{A}$ has real rank zero.

Proposition 13. *The real λ-function on the self-adjoint part of the unit ball $\mathcal{A}^1_{\rm sa}$ of a unital C^*-algebra $\mathcal{A}$ is given by the following formulae:*

$$\lambda_r(x) = \tfrac{1}{2}(1 + m(x)) \qquad \text{if} \quad x \in \mathcal{A}^{-1}_{\rm sa} \tag{3}$$

$$\lambda_r(x) = \tfrac{1}{2}(1 - \alpha_r(x)) \qquad \text{if} \quad x \notin \mathcal{A}^{-1}_{\rm sa}. \tag{4}$$

Proof. If $x \in \mathcal{A}^{-1}_{\rm sa}$ with polar decomposition $x = u|x|$ then with $\lambda = (1 + m(x))/2$ we define the element $y = (1-\lambda)^{-1}(x - \lambda u)$ in $\mathcal{A}_{\rm sa}$. Using Definition 2 it follows by easy computations in spectral theory that $\|y\| \leq 1$. Thus $x = \lambda u + (1 - \lambda y)$ in $\mathcal{A}^1_{\rm sa}$, whence $\lambda_r(x) \geq (1 + m(x))/2$.

Conversely, if $x = \lambda w + (1-\lambda)z$ for some w in $\mathcal{S}(\mathcal{A})$ and z in $\mathcal{A}^1_{\rm sa}$ then by Proposition 12

$$1 - m(x) \leq \|w - x\| = \|(1-\lambda)(w - z)\| \leq 2(1-\lambda),$$

whence $\lambda \leq (1 + m(x))/2$, as desired.

If $x \notin \mathcal{A}^{-1}_{\rm sa}$ and $x = \lambda w + (1-\lambda)z$ for some w in $\mathcal{S}(\mathcal{A})$ and z in $\mathcal{A}^1_{\rm sa}$ then by Proposition 10

$$1 + \alpha_r(x) \leq \|w - x\| = \|(1-\lambda)(w - z)\| \leq 2(1-\lambda),$$

whence $\lambda \leq (1 - \alpha_r(x))/2$, which is therefore an upper bound for $\lambda_r(x)$.

On the other hand, if $\alpha_r(x) \neq 1$ and $\alpha_r(x) < \delta < 1$ we can by Theorem 3 find a projection p in $\mathcal{A}$ with $p_\delta \leq p \leq \mathbf{1} - q_\delta$. With $u = 2p - \mathbf{1}$ and $\lambda = (1-\delta)/2$ we claim that the element $y = (1-\lambda)^{-1}(x - \lambda u)$ has norm at most one. To prove this we compute

$$\|yp_\delta\| = \|(1-\lambda)^{-1}(x-\lambda)p_\delta\| \leq 1.$$

Similarly $\|yq_\delta\| \leq 1$. Finally,

$$\|y(\mathbf{1} - p_\delta - q_\delta)\| \leq \|(1-\lambda)^{-1}(\delta + \lambda)\| \leq 1.$$

Consequently $\|y\| \leq 1$. Since $x = \lambda u + (1-\lambda)y$ by construction, we see that $\lambda_r(x) \geq \lambda$, whence in the limit as $\delta \to \alpha_r(x)$ we obtain the desired estimate $\lambda_r(x) \geq (1 - \alpha_r(x))/2$. □

Remark 14. We see from the formulae in Proposition 13 that if $\lambda_r(x) > 0$ for every x in $\mathcal{A}^1_{\text{sa}}$ then $\alpha_r(x) < 1$ for every x. But if $\alpha_r(x) > 0$ for some element x in $\mathcal{A}^1_{\text{sa}}$ then by Corollary 4 we have a non-zero element x_α in $\mathcal{A}_{\text{sa}}$ with $\|x_\alpha\| = \alpha_r(x_\alpha)$. Thus the element $y = \|x_\alpha\|^{-1}x_\alpha$ will violate the real λ-condition ($\lambda_r(y) = 0$). The only way to avoid this situation is to demand that $\alpha_r(x) = 0$ for all x, so that $\mathcal{A}$ has real rank zero. In this case, of course, $\lambda_r(x) \geq 1/2$ for every x in $\mathcal{A}^1_{\text{sa}}$.

3.3 Projectionless C^*-Algebras

It is well known that there are C^*-algebras, even simple ones, that contain no non-trivial projections. Such algebras may be regarded as opposite to the real rank zero C^*-algebras. Theorem 3 allows us to reformulate this property in terms of the distance from the invertible self-adjoint elements in the algebra.

Proposition 15. *In a unital C^*-algebra $\mathcal{A}$ the following conditions are equivalent:*

(i) *$\mathcal{A}$ has no non-trivial projections.*
(ii) $\alpha_r(x) = \min\{\|x_+\|,\ \|x_-\|\}$ *for every element x in $\mathcal{A}_{\text{sa}}$.*
(iii) $\mathcal{A}^{-1}_{\text{sa}} \subset -\mathcal{A}_+ \cup \mathcal{A}_+$.

Proof. (i) $\Longrightarrow$ (ii) If we can find δ such that $\alpha_r(x) < \delta < \min\{\|x_+\|,\ \|x_-\|\}$ for some x in $\mathcal{A}_{\text{sa}}$, then the spectral projections p_δ and q_δ are both non-zero (in $\mathcal{A}^{**}$). Applying Theorem 3 we obtain a projection p in $\mathcal{A}$ such that $p_\delta \leq p \leq \mathbf{1} - q_\delta$, which means that p is non-trivial.

(ii) $\Longrightarrow$ (iii) We always have $-\mathcal{A}_+ \cup \mathcal{A}_+ \subset (\mathcal{A}^{-1}_{\text{sa}})^=$ and $\alpha_r(x) \leq \min\{\|x_+\|,\ \|x_-\|\}$. If now $x \in (\mathcal{A}^{-1}_{\text{sa}})^= \setminus (-\mathcal{A}_+ \cup \mathcal{A}_+)$ then $\alpha_r(x) = 0$, but $\min\{\|x_+\|,\ \|x_-\|\} > 0$.

(iii) $\Longrightarrow$ (i) If p is a non-trivial projection in $\mathcal{A}$ then $2p - \mathbf{1} \in \mathcal{A}^{-1}_{\text{sa}} \setminus (-\mathcal{A}_+ \cup \mathcal{A}_+)$. □

3.4 The Unitary Case Revisited

It is well worth noticing that the method from Theorem 3 also can be used to give a short and transparent proof of the result mentioned in 1.1 about unitary polar decomposition of arbitrary elements in a C^*-algebra $\mathcal{A}$. With some more work it will even give the corresponding result in [4, Theorem 2.2] for quasi-invertible elements, but we shall here only show the former.

If $x \in \mathcal{A}$ with polar decomposition $x = v|x|$ we define the operator matrix

$$a = \begin{pmatrix} vc_\delta(|x|) & d_\delta(|x^*|) \\ d_\delta(|x|) & -x^* \end{pmatrix}$$

with c_δ and d_δ as in Lemma 1. The observant reader will notice the similarity between a and the standard unitary dilation of a contraction x. Straightforward computations show that

$$a^*a = \begin{pmatrix} \delta^2\mathbf{1} & 0 \\ 0 & \delta^2\mathbf{1} \vee xx^* \end{pmatrix} \quad \text{and} \quad aa^* = \begin{pmatrix} \delta^2\mathbf{1} & 0 \\ 0 & \delta^2\mathbf{1} \vee x^*x \end{pmatrix},$$

so that $m(a) \geq \delta$ and $m(a^*) \geq \delta$. If therefore $\alpha(x) < \delta$, so that we can find y in $\mathcal{A}^{-1}$ with $\|x^* - y\| < \delta$, then the matrix

$$b = \begin{pmatrix} vc_\delta(|x|) & d_\delta(|x^*|) \\ d_\delta(|x|) & -y \end{pmatrix}$$

is invertible (in $\mathbb{M}_2(\mathcal{A})$), since both $m(b) > 0$ and $m(b^*) > 0$, cf. Definition 2. As in the proof of Theorem 3 this implies that also the element $z = vc_\delta(|x|) + d_\delta(|x^*|)y^{-1}d_\delta(|x|)$ is invertible (in $\mathcal{A}$). Since $ze_\delta = \delta ve_\delta$ and $f_\delta z = \delta f_\delta v$ by construction, it follows that if $z = u|z|$ is the polar decomposition of z then u is a unitary in $\mathcal{A}$ such that $ue_\delta = ve_\delta$ and $f_\delta u = f_\delta v$. We have reproved [19, Theorem 2.2] and [15, Theorem 5]

Theorem 16. *If $x = v|x|$ is the polar decomposition of an element in a unital C^*-algebra $\mathcal{A}$ then for each $\delta > \alpha(x)$ there is a unitary u in $\mathcal{A}$ such that $ue_\delta = ve_\delta$. Equivalently, for every continuous function f vanishing on $[0, \delta]$ we have $vf(|x|) = uf(|x|)$.* □

References

1. Blackadar, B.: Notes on the structure of projections in simple C^*-algebras. Semesterbericht Funktionalanalysis, W82, Universität Tübingen, 1983.
2. Brown, L.G.: Interpolation by projections in C^*-algebras of real rank zero. Journal of Operator Theory, **26**, 383–387, 1991
3. Brown, L.G., Pedersen, G.K.: C^*-algebras of real rank zero. Journal of Functional Analysis **99**, 131–149, 1991.
4. Brown, L.G., Pedersen, G.K.: On the geometry of the unit ball of a C^*-algebra. Journal für die reine und angewandte Mathematik **469**, 113–147, 1995

5. Brown, L.G., Pedersen, G.K.: Approximation and convex decomposition by extremals in a C^*-algebra. Mathematica Scandinavica, **81**, 69–85, 1997.
6. Brown, L.G., Pedersen, G.K.: Ideal structure and C^*-algebras of low rank. Mathematica Scandinavica, to appear.
7. Brown, L.G., Pedersen, G.K.: Limits and C^*-algebras of low rank or dimension.
8. Haagerup, U., Rørdam, M.: C^*-algebras of unitary rank two. Journal of Operator Theory, **30**, 161–171, 1993
9. Leichtnam, E., Lück, W.: On the cut and paste property of higher signatures of a closed oriented manifold. Topology, **41**, 725–744, 2002.
10. Leichtnam, E., Piazza, P.: Spectral sections and higher Atiyah–Patodi–Singer index theory on Galois coverings. Geometric and Functional Analysis, **8**, 171–189, 1998.
11. Leichtnam, E., Piazza, P.: Dirac index classes and the noncommutative spectral flow. Journal of Functional Analysis, **200**, 348–400, 2003.
12. Leichtnam, E., Lott, J., Piazza, P.: On the homotopy invariance of higher signatures for manifolds with boundary. Journal of Differential Geometry, **54**, 561–633, 2000.
13. Lott, J.: The zero-in-the-spectrum question. L'Enseignement Mathématique, **42**, 341–376, 1996.
14. Olsen, C.L.: Unitary approximation. Journal of Functional Analysis, **85**, 392–419, 1989.
15. Pedersen, G.K.: Unitary extensions and polar decompositions in a C^*-algebra. Journal of Operator Theory, **17**, 357–364, 1987.
16. Pedersen, G.K.: Three quavers on unitary elements in C^*-algebras. Pacific Journal of Mathematics, **137**, 169–180, 1989.
17. Pedersen, G.K.: The λ-function in operator algebras. Journal of Operator Theory **26**, 345–381, 1991.
18. Rieffel, M.A.: Dimension and stable rank in the K-theory of C^*-algebras. Proceedings of the London Mathematical Society (3) **46**, 301–333, 1983.
19. Rørdam, M.: Advances in the theory of unitary rank and regular approximation. Annals of Mathematics **128**, 153–172, 1988.

KMS States and Complex Multiplication (Part II*)

Alain Connes[1], Matilde Marcolli[2], and Niranjan Ramachandran[3]

[1] Collège de France, 3, rue d'Ulm, Paris, F-75005 France, alain@connes.org
[2] Max–Planck Institut für Mathematik, Vivatsgasse 7, Bonn, D-53111 Germany, marcolli@mpim-bonn.mpg.de
[3] Department of Mathematics, University of Maryland, College Park, MD 20912 USA, atma@math.umd.edu

1 Introduction

Several results point to deep relations between noncommutative geometry and class field theory ([3], [10], [20], [22]). In [3] a quantum statistical mechanical system (BC) is exhibited, with partition function the Riemann zeta function $\zeta(\beta)$, and whose arithmetic properties are related to the Galois theory of the maximal abelian extension of $\mathbb{Q}$. In [10], this system is reinterpreted in terms of the geometry of commensurable 1-dimensional $\mathbb{Q}$-lattices, and a generalization is constructed for 2-dimensional $\mathbb{Q}$-lattices. The arithmetic properties of this GL_2-system and its extremal KMS states at zero temperature are related to the Galois theory of the modular field F, that is, the field of elliptic modular functions. These are functions on modular curves, *i.e.* on moduli spaces of elliptic curves. The low temperature extremal KMS states and the Galois properties of the GL_2-system are analyzed in [10] for the generic case of elliptic curves with transcendental j-invariant. As the results of [10] show, one of the main new features of the GL_2-system is the presence of symmetries by *endomorphism*, as in (8) below. The full Galois group of the modular field appears then as symmetries, acting on the set of extremal KMS_β states of the system, for large inverse temperature β.

In both the original BC system and in the GL_2-system, the arithmetic properties of zero temperature KMS states rely on an underlying result of compatibility between adèlic groups of symmetries and Galois groups. This correspondence between adèlic and Galois groups naturally arises within the context of Shimura varieties. In fact, a Shimura variety is a pro-variety defined over $\mathbb{Q}$, with a rich adèlic group of symmetries. In that context, the compatibility of the Galois action and the automorphisms is at the heart of Langlands program. This leads us to give a reinterpretation of the BC and the GL_2

*Part I is the publication listed as reference [11] systems in the language of Shimura varieties, with the BC system corresponding to the simplest (zero dimensional) Shimura variety $Sh(\mathrm{GL}_1, \pm 1)$. In the case of the GL_2 system, we show how the data of 2-dimensional $\mathbb{Q}$-lattices and commensurability can be also described in terms of elliptic curves together with a pair of points in the total Tate module, and the system is related to the Shimura variety $Sh(\mathrm{GL}_2, \mathbb{H}^{\pm})$ of GL_2. This viewpoint suggests considering our systems as *noncommutative pro-varieties* defined over $\mathbb{Q}$, more specifically as noncommutative Shimura varieties.

This point of view, which we discuss here only in the simplest case of $G = \mathrm{GL}_1$ and $G = \mathrm{GL}_2$ was extended to a wide class of Shimura varieties by E. Ha and F. Paugam, [13]. They constructed generalizations of the BC and GL_2 system that provide noncommutative Shimura varieties, and investigated the arithmetic properties of their partition functions and KMS states.

In our paper [11], we construct a quantum statistical mechanical system associated to an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$, $d > 0$ a positive integer. Just like the BC and the GL_2 systems are based on the geometric notion of $\mathbb{Q}$-lattices and commensurability, this "complex multiplication system" (CM) is based on an analogous geometric notion of commensurability of 1-dimensional K-lattices.

The arithmetic properties of the CM system fully incorporate the explicit class field theory for the imaginary quadratic field K, and its partition function is the Dedekind zeta function $\zeta_K(\beta)$ of K. Thus, the main result of [11], which we recall here in Theorem 24 below, gives a complete answer, in the case of an imaginary quadratic field K, to the following question, which has been open since the work of Bost and Connes [3].

Problem 1. For some number field K (other than $\mathbb{Q}$) exhibit an explicit quantum statistical mechanical system $(\mathcal{A}, \sigma_t)$ with the following properties:

1. The partition function $Z(\beta)$ is the Dedekind zeta function of K.
2. The system has a phase transition with spontaneous symmetry breaking at the pole $\beta = 1$ of the zeta function.
3. There is a unique equilibrium state above critical temperature.
4. The quotient C_K/D_K of the idèles class group of K by the connected component of the identity acts as symmetries of the system $(\mathcal{A}, \sigma_t)$.
5. There is a subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ with the property that the values of extremal ground states on elements of $\mathcal{A}_0$ are algebraic numbers and generate the maximal abelian extension K^{ab}.
6. The Galois action on these values is realized by the induced action of C_K/D_K on the ground states, via the class field theory isomorphism $\theta : C_K/D_K \to \mathrm{Gal}(K^{ab}/K)$.

The BC system satisfies all the properties listed in Problem 1, in the case of $K = \mathbb{Q}$. It is natural, therefore, to pose the analogous question in the case of other number fields. Some important progress in the direction of generalizing the BC system to other number fields was done by Harari and Leichtnam [16], Cohen [5], Arledge, Laca and Raeburn [1], Laca and van Frankenhuijsen [20]. However, to our knowledge, the first construction of a system satisfying all the conditions posed in Problem 1, with no restriction on the class number of K, was the one obtained in [11]. As we shall discuss at length in the present paper, one reason why progress in the solution of Problem 1 is difficult is that the requirement on the Galois action on ground states (zero temperature KMS states) of the system is very closely related to a difficult problem in number theory, namely Hilbert's 12th problem on explicit class field theory. We will argue in this paper that Problem 1 may provide a new possible approach to Hilbert's 12th problem.

In this perspective, the BC system and the CM system of [11] cover the two known cases ($K = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{-d})$) of Hilbert's 12th problem. We analyze closely the properties of both the BC and the CM system, in order to understand what new insight they may give on this problem and in what part instead they depend on the number theoretic solution.

The new CM system we constructed in [11] can be regarded in two different ways. On the one hand, it is a generalization of the BC system of [3], when changing the field from $\mathbb{Q}$ to $K = \mathbb{Q}(\sqrt{-d})$, and is in fact Morita equivalent to the one considered in [20], but with no restriction on the class number. On the other hand, it is also a specialization of the GL_2-system of [10] to elliptic curves with complex multiplication by K. The KMS_∞ states of the CM system can be related to the non-generic KMS_∞ states of the GL_2-system, associated to points $\tau \in \mathbb{H}$ with complex multiplication by K, and the group of symmetries is the Galois group of the maximal abelian extension of K.

Here also symmetries by endomorphisms play a crucial role, as they allow for the action of the class group $\mathrm{Cl}(\mathcal{O})$ of the ring $\mathcal{O}$ of algebraic integers of K, so that the properties of Problem 1 are satisfied in all cases, with no restriction on the class number of K.

As we showed in [11], the CM system can be realized as a subgroupoid of the GL_2-system. It has then a natural choice of an arithmetic subalgebra inherited from that of the GL_2-system. This is crucial, in order to obtain the intertwining of Galois action on the values of extremal KMS states and action of symmetries of the system.

The paper is structured as follows. In Sections 2 and 3 we discuss the relation between Problem 1 and Hilbert's 12th problem and the geometry of the BC and GL_2 system from the point of view of Shimura varieties. In Section 4 we recall the construction and main properties of the CM system, especially its relation to the explicit class field theory for imaginary quadratic fields. We also compare it with the BC and GL_2 systems and with previous systems introduced as generalizations of the BC system.

We summarize and compare the main properties of the three systems (BC, GL_2, and CM) in the following table.

System	GL_1	GL_2	CM
Partition function	$\zeta(\beta)$	$\zeta(\beta)\zeta(\beta-1)$	$\zeta_K(\beta)$
Symmetries	$\mathbb{A}_f^*/\mathbb{Q}^*$	$GL_2(\mathbb{A}_f)/\mathbb{Q}^*$	$\mathbb{A}_{K,f}^*/K^*$
Symmetry group	Compact	Locally compact	Compact
Automorphisms	$\hat{\mathbb{Z}}^*$	$GL_2(\hat{\mathbb{Z}})$	$\hat{\mathcal{O}}^*/\mathcal{O}^*$
Endomorphisms		$GL_2^+(\mathbb{Q})$	$Cl(\mathcal{O})$
Galois group	$Gal(\mathbb{Q}^{ab}/\mathbb{Q})$	$Aut(F)$	$Gal(K^{ab}/K)$
Extremal KMS_∞	$Sh(GL_1, \pm 1)$	$Sh(GL_2, \mathbb{H}^\pm)$	$\mathbb{A}_{K,f}^*/K^*$

1.1 Notation

In the table above and in the rest of the paper, we denote by $\hat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$ and by $\mathbb{A}_f = \hat{\mathbb{Z}} \otimes \mathbb{Q}$ the ring of finite adeles of $\mathbb{Q}$. For any abelian group G, we denote by G_{tors} the subgroup of elements of finite order. For any ring R, we write R^* for the group of invertible elements, while $R^\times$ denotes the set of nonzero elements of R, which is a semigroup if R is an integral domain. We write $\mathcal{O}$ for the ring of algebraic integers of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$, where d is a positive integer. We also use the notation

$$\hat{\mathcal{O}} := \mathcal{O} \otimes \hat{\mathbb{Z}} \quad \mathbb{A}_{K,f} = \mathbb{A}_f \otimes_{\mathbb{Q}} K \quad \text{and} \quad \mathbb{I}_K = \mathbb{A}_{K,f}^* = GL_1(\mathbb{A}_{K,f}). \tag{1}$$

Notice that K^* embeds diagonally into $\mathbb{I}_K$.

The modular field F is the field of modular functions over $\mathbb{Q}^{ab}$ (*cf. e.g.* [21]). This is the union of the fields F_N of modular functions of level N rational over the cyclotomic field $\mathbb{Q}(\zeta_N)$, that is, such that the q-expansion at a cusp has coefficients in the cyclotomic field $\mathbb{Q}(\zeta_N)$.

G. Shimura determined the automorphisms of F (*cf.* [32]). His result

$$GL_2(\mathbb{A}_f)/\mathbb{Q}^* \xrightarrow{\sim} Aut(F),$$

is a non-commutative analogue of the class field theory isomorphism which provides the canonical identifications

$$\theta : \mathbb{I}_K/K^* \xrightarrow{\sim} Gal(K^{ab}/K)\,, \tag{2}$$

and $\mathbb{A}_f^*/\mathbb{Q}_+^* \xrightarrow{\sim} Gal(\mathbb{Q}^{ab}/\mathbb{Q})$.

2 Quantum Statistical Mechanics and Explicit Class Field Theory

The BC quantum statistical mechanical system [2, 3] exhibits generators of the maximal abelian extension of $\mathbb{Q}$, parameterizing extremal zero temperature states. Moreover, the system has the remarkable property that extremal KMS_∞ states take algebraic values, when evaluated on a rational subalgebra of the C^*-algebra of observables. The action on these values of the absolute Galois group factors through the abelianization $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ and is implemented by the action of the idèle class group as symmetries of the system, via the class field theory isomorphism. This suggests the intriguing possibility of using the setting of quantum statistical mechanics to address the problem of explicit class field theory for other number fields.

In this section we recall some basic notions of quantum statistical mechanics and of class field theory, which will be used throughout the paper. We also formulate a general conjectural relation between quantum statistical mechanics and the explicit class field theory problem for number fields.

2.1 Quantum Statistical Mechanics

A quantum statistical mechanical system consists of an algebra of observables, given by a unital C^*-algebra $\mathcal{A}$, together with a time evolution, consisting of a 1-parameter group of automorphisms σ_t, $(t \in \mathbb{R})$, whose infinitesimal generator is the Hamiltonian of the system, $\sigma_t(x) = e^{itH} x e^{-itH}$. The analog of a probability measure, assigning to every observable a certain average, is given by a state, namely a continuous linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ satisfying positivity, $\varphi(x^*x) \geq 0$, for all $x \in \mathcal{A}$, and normalization, $\varphi(1) = 1$. In the quantum mechanical framework, the analog of the classical Gibbs measure is given by states satisfying the KMS condition (*cf.* [15]).

Definition 2. *A triple $(\mathcal{A}, \sigma_t, \varphi)$ satisfies the Kubo-Martin-Schwinger (KMS) condition at inverse temperature $0 \leq \beta < \infty$, if the following holds. For all $x, y \in \mathcal{A}$, there exists a holomorphic function $F_{x,y}(z)$ on the strip $0 < \mathrm{Im}(z) < \beta$, which extends as a continuous function on the boundary of the strip, with the property that*

$$F_{x,y}(t) = \varphi(x\sigma_t(y)) \quad \text{and} \quad F_{x,y}(t + i\beta) = \varphi(\sigma_t(y)x), \quad \forall t \in \mathbb{R}. \tag{3}$$

We also say that φ is a KMS_β state for $(\mathcal{A}, \sigma_t)$. The set $\mathcal{K}_\beta$ of KMS_β states is a compact convex Choquet simplex [4, II §5] whose set of extreme points $\mathcal{E}_\beta$ consists of the factor states. One can express any KMS_β state uniquely in terms of extremal states, because of the uniqueness of the barycentric decomposition of a Choquet simplex.

At 0 temperature ($\beta = \infty$) the KMS condition (3) says that, for all $x, y \in \mathcal{A}$, the function

$$F_{x,y}(t) = \varphi(x\,\sigma_t(y)) \tag{4}$$

extends to a bounded holomorphic function in the upper half plane $\mathbb{H}$. This implies that, in the Hilbert space of the GNS representation of φ (*i.e.* the completion of $\mathcal{A}$ in the inner product $\varphi(x^*y)$), the generator H of the one-parameter group σ_t is a positive operator (positive energy condition). However, this notion of 0-temperature KMS states is in general too weak, hence the notion of KMS_∞ states that we shall consider is the following.

Definition 3. *A state φ is a KMS_∞ state for $(\mathcal{A}, \sigma_t)$ if it is a weak limit of KMS_β states, that is, $\varphi_\infty(a) = \lim_{\beta\to\infty} \varphi_\beta(a)$ for all $a \in \mathcal{A}$.*

One can easily see the difference between the KMS condition at zero tempretature and the notion of KMS_∞ states given in Definition 3, in the simple case of the trivial time evolution $\sigma_t = \mathrm{id}\,, \quad \forall t \in \mathbb{R}$. In this case, any state has the property that (4) extends to the upper half plane (as a constant). On the other hand, only tracial states can be weak limits of β-KMS states, hence the notion given in Definition 3 is more restrictive. With Definition 3 we still obtain a weakly compact convex set Σ_∞ and we can consider the set $\mathcal{E}_\infty$ of its extremal points.

The typical framework for spontaneous symmetry breaking in a system with a unique phase transition (*cf.* [14]) is that the simplex Σ_β consists of a single point for $\beta \leqq \beta_c$ *i.e.* when the temperature is larger than the critical temperature T_c, and is non-trivial (of some higher dimension in general) when the temperature lowers. A (compact) group of automorphisms $G \subset \mathrm{Aut}(\mathcal{A})$ commuting with the time evolution,

$$\sigma_t\,\alpha_g = \alpha_g\,\sigma_t \qquad \forall\, g \in G\,,\ t \in \mathbb{R}\,, \tag{5}$$

is a symmetry group of the system. Such G acts on Σ_β for any β, hence on the set of extreme points $\mathcal{E}(\Sigma_\beta) = \mathcal{E}_\beta$. The choice of an equilibrium state $\varphi \in \mathcal{E}_\beta$ may break this symmetry to a smaller subgroup given by the isotropy group $G_\varphi = \{g \in G\,,\ g\varphi = \varphi\}$.

The unitary group $\mathcal{U}$ of the fixed point algebra of σ_t acts by inner automorphisms of the dynamical system $(\mathcal{A}, \sigma_t)$, by

$$(\mathrm{Ad}u)\,(a) := \,u\,a\,u^*\,, \quad \forall a \in \mathcal{A}\,, \tag{6}$$

for all $u \in \mathcal{U}$. One can define an action *modulo inner* of a group G on the system $(\mathcal{A}, \sigma_t)$ as a map $\alpha : G \to \mathrm{Aut}(\mathcal{A}, \sigma_t)$ fulfilling the condition

$$\alpha(gh)\,\alpha(h)^{-1}\,\alpha(g)^{-1} \in \mathrm{Inn}(\mathcal{A}, \sigma_t)\,, \quad \forall g, h \in G\,, \tag{7}$$

i.e. , as a homomorphism $\alpha : G \to \mathrm{Aut}(A, \sigma_t)/\mathcal{U}$. The KMS_β condition shows that the *inner* automorphisms $\mathrm{Inn}(\mathcal{A}, \sigma_t)$ act trivially on KMS_β states, hence (7) induces an action of the group G on the set Σ_β of KMS_β states, for $0 < \beta \leq \infty$.

More generally, one can consider actions *by endomorphisms* (*cf.* [10]), where an endomorphism ρ of the dynamical system $(\mathcal{A}, \sigma_t)$ is a $*$-homomorphism $\rho : \mathcal{A} \to \mathcal{A}$ commuting with the evolution σ_t. There is an induced action of ρ on KMS$_\beta$ states, for $0 < \beta < \infty$, given by

$$\rho^*(\varphi) := \frac{\varphi \circ \rho}{\varphi(\rho(1))}, \tag{8}$$

provided that $\varphi(\rho(1)) \neq 0$, where $\rho(1)$ is an idempotent fixed by σ_t.

An *isometry* $u \in \mathcal{A}$, $u^* u = 1$, satisfying $\sigma_t(u) = \lambda^{it} u$ for all $t \in \mathbb{R}$ and for some $\lambda \in \mathbb{R}_+^*$, defines an *inner* endomorphism Adu of the dynamical system $(\mathcal{A}, \sigma_t)$, again of the form (6). The KMS$_\beta$ condition shows that the induced action of Adu on Σ_β is trivial, *cf.* [10].

In general, the induced action (modulo inner) of a semigroup of endomorphisms of $(\mathcal{A}, \sigma_t)$ on the KMS$_\beta$ states need not extend directly (in a nontrivial way) to KMS$_\infty$ states. In fact, even though (8) is defined for states $\varphi_\beta \in \mathcal{E}_\beta$, it can happen that, when passing to the weak limit $\varphi = \lim_\beta \varphi_\beta$ one has $\varphi(\rho(1)) = 0$ and can no longer apply (8).

In such cases, it is often still possible to obtain an induced nontrivial action on $\mathcal{E}_\infty$. This can be done via the following procedure, which was named "warming up and cooling down" in [10]. One considers first a map $W_\beta : \mathcal{E}_\infty \to \mathcal{E}_\beta$ (called the "warming up" map) given by

$$W_\beta(\varphi)(a) = \frac{\mathrm{Tr}(\pi_\varphi(a)\, e^{-\beta H})}{\mathrm{Tr}(\, e^{-\beta H})}, \quad \forall a \in \mathcal{A}\,. \tag{9}$$

Here H is the positive energy Hamiltonian, implementing the time evolution in the GNS representation π_φ associated to the extremal KMS$_\infty$ state φ. Assume that, for sufficiently large β, the map (9) gives a bijection between KMS$_\infty$ states (in the sense of Definition 3) and KMS$_\beta$ states. The action by endomorphisms on $\mathcal{E}_\infty$ is then defined as

$$\rho^*(\varphi)(a) := \lim_{\beta \to \infty} \rho^*(W_\beta(\varphi))(a) \qquad \forall a \in \mathcal{A}. \tag{10}$$

This type of symmetries, implemented by endomorphisms instead of automorphisms, plays a crucial role in the theory of superselection sectors in quantum field theory, developed by Doplicher–Haag–Roberts (cf.[14], Chapter IV).

States on a C^*-algebra extend the notion of integration with respect to a measure in the commutative case. In the case of a non-unital algebra, the multipliers algebra provides a compactification, which corresponds to the Stone–Čech compactification in the commutative case. A state admits a canonical extension to the multiplier algebra. Moreover, just as in the commutative case one can extend integration to certain classes of unbounded functions, it is preferable to extend, whenever possible, the integration provided by a state to certain classes of unbounded multipliers.

2.2 Hilbert's 12th Problem

The main theorem of class field theory provides a classification of finite abelian extensions of a local or global field K in terms of subgroups of a locally compact abelian group canonically associated to the field. This is the multiplicative group $K^* = \mathrm{GL}_1(K)$ in the local non-archimedean case, while in the global case it is the quotient of the idèle class group C_K by the connected component of the identity. The construction of the group C_K is at the origin of the theory of idèles and adèles.

Hilbert's 12th problem can be formulated as the question of providing an explicit description of a set of generators of the maximal abelian extension K^{ab} of a number field K and an explicit description of the action of the Galois group $\mathrm{Gal}(K^{ab}/K)$ on them. This Galois group is the maximal abelian quotient of the absolute Galois group $\mathrm{Gal}(\bar{K}/K)$ of K, where $\bar{K}$ denotes an algebraic closure of K.

Remarkably, the only cases of number fields for which there is a complete answer to Hilbert's 12th problem are the construction of the maximal abelian extension of $\mathbb{Q}$ using torsion points of $\mathbb{C}^*$ (Kronecker–Weber) and the case of imaginary quadratic fields, where the construction relies on the theory of elliptic curves with complex multiplication (*cf. e.g.* the survey [33]).

If $\mathbb{A}_K$ denotes the adèles of a number field K and $J_K = \mathrm{GL}_1(\mathbb{A}_K)$ is the group of idèles of K, we write C_K for the group of idèle classes $C_K = J_K/K^*$ and D_K for the connected component of the identity in C_K.

2.3 Fabulous States for Number Fields

We discuss here briefly the relation of Problem 1 to Hilbert's 12th problem, by concentrating on the arithmetic properties of the action of symmetries on the set $\mathcal{E}_\infty$ of extremal zero temperature KMS states. We abstract these properties in the notion of "fabulous states" discussed below.

Given a number field K, with a choice of an embedding $K \subset \mathbb{C}$, the "problem of fabulous states" consists of the following question.

Problem 4. Construct a C^*-dynamical system $(\mathcal{A}, \sigma_t)$, with an *arithmetic subalgebra* $\mathcal{A}_K$ of $\mathcal{A}$, with the following properties:

1. The quotient group $G = C_K/D_K$ acts on $\mathcal{A}$ as symmetries compatible with σ_t and preserving $\mathcal{A}_K$.
2. The states $\varphi \in \mathcal{E}_\infty$, evaluated on elements of the arithmetic subalgebra $\mathcal{A}_K$, satisfy:
   - $\varphi(a) \in \overline{K}$, the algebraic closure of K in $\mathbb{C}$;
   - the elements of $\{\varphi(a) : \ a \in \mathcal{A}_K, \ \varphi \in \mathcal{E}_\infty\}$ generate K^{ab}.
3. The class field theory isomorphism

$$\theta : C_K/D_K \xrightarrow{\simeq} \mathrm{Gal}(K^{ab}/K) \tag{11}$$

intertwines the actions,

$$\alpha(\varphi(a)) = (\varphi \circ \theta^{-1}(\alpha))(a), \tag{12}$$

for all $\alpha \in \mathrm{Gal}(K^{ab}/K)$, for all $\varphi \in \mathcal{E}_\infty$, and for all $a \in \mathcal{A}_K$.

4. The algebra $\mathcal{A}_K$ has an explicit presentation by generators and relations.

It is important to make a general remark about Problem 4 versus the original Hilbert's 12th problem. It may seem at first that the formulation of Problem 4 will not lead to any new information about the Hilbert 12th problem. In fact, one can always consider here a "trivial system" where $\mathcal{A} = C(C_K/D_K)$ and $\sigma_t = id$. All states are KMS states and the extremal ones are just valuations at points. Then one can always choose as the "rational subalgebra" the field K^{ab} itself, embedded in the space of smooth functions on C_K/D_K as $x \mapsto f_x(g) = \theta(g)(x)$, for $x \in K^{ab}$. Then this trivial system tautologically satisfies all the properties of Problem 4, except the last one. This last property, for this system, is then exactly as difficult as the original Hilbert 12th problem. This appears to show that the construction of "fabulous states" is just a reformulation of the original problem and need not simplify the task of obtaining explicit information about the generators of K^{ab} and the Galois action.

The point is precisely that such "trivial example" is only a reformulation of the Hilbert 12th problem, while Problem 4 allows for *nontrivial* constructions of quantum statistical mechanical systems, where one can use essentially the fact of working with algebras instead of fields. The hope is that, for suitable systems, giving a presentation of the algebra $\mathcal{A}_K$ will prove to be an easier problem than giving generators of the field K^{ab} and similarly for the action by symmetries on the algebra as opposed to the Galois action on the field.

Solutions to Problem 1 give *nontrivial* quantum statistical mechanical systems with the right Galois symmetries and arithmetic properties of zero temperature KMS states. (Notice that the trivial example above certainly will not satisfy the other properties sought for in Problem 1.) The hope is that, for such systems, the problem of giving a presentation of the algebra $\mathcal{A}_K$ may turn out to be easier than the original Hilbert 12th problem. We show in Proposition 7 in Section 3.3 below how, in the case of $K = \mathbb{Q}$, using the algebra of the BC system of [3], one can obtain a very simple description of $\mathbb{Q}^{ab}$ that uses only the explicit presentation of the algebra, without any reference to field extensions. This is possible precisely because the algebra is larger than the part that corresponds to the trivial system.

A broader type of question, in a similar spirit, can be formulated regarding the construction of quantum statistical mechanical systems with adèlic groups of symmetries and the arithmetic properties of its action on zero temperature extremal KMS states. The case of the GL_2-system of [10] fits into this general program.

2.4 Noncommutative Pro-Varieties

In the setting above, the C^*-dynamical system $(\mathcal{A}, \sigma_t)$ together with a $\mathbb{Q}$-structure compatible with the flow σ_t (*i.e.* a rational subalgebra $\mathcal{A}_{\mathbb{Q}} \subset \mathcal{A}$ such that $\sigma_t(\mathcal{A}_{\mathbb{Q}} \otimes \mathbb{C}) = \mathcal{A}_{\mathbb{Q}} \otimes \mathbb{C}$) defines a *non-commutative algebraic (pro-)variety* X over $\mathbb{Q}$. The ring $\mathcal{A}_{\mathbb{Q}}$ (or $\mathcal{A}_{\mathbb{Q}} \otimes \mathbb{C}$), which need not be involutive, is the analog of the ring of algebraic functions on X and the set of extremal KMS_∞-states is the analog of the set of points of X. As we will see for example in (15) and (43) below, the action of the subgroup of $\mathrm{Aut}(\mathcal{A}, \sigma_t)$ which takes $\mathcal{A}_{\mathbb{Q}} \otimes \mathbb{C}$ into itself is analogous to the action of the Galois group on the (algebraic) values of algebraic functions at points of X.

The analogy illustrated above leads us to speak somewhat loosely of "classical points" of such a noncommutative algebraic pro-variety. We do not attempt to give a general definition of classical points, while we simply remark that, for the specific construction considered here, such a notion is provided by the zero temperature extremal states.

3 $\mathbb{Q}$-lattices and Noncommutative Shimura Varieties

In this section we recall the main properties of the BC and the GL_2 system, which will be useful for our main result.

Both cases can be formulated starting with the same geometric notion, that of commensurability classes of $\mathbb{Q}$-lattices, in dimension one and two, respectively.

Definition 5. *A $\mathbb{Q}$-lattice in $\mathbb{R}^n$ is a pair (Λ, ϕ), with Λ a lattice in $\mathbb{R}^n$, and*

$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda \tag{13}$$

a homomorphism of abelian groups. A $\mathbb{Q}$-lattice is invertible if the map (13) *is an isomorphism. Two $\mathbb{Q}$-lattices (Λ_1, ϕ_1) and (Λ_2, ϕ_2) are commensurable if the lattices are commensurable (i.e. $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$) and the maps agree modulo the sum of the lattices,*

$$\phi_1 \equiv \phi_2 \mod \Lambda_1 + \Lambda_2 .$$

It is essential here that one does not require the homomorphism ϕ to be invertible in general.

The set of $\mathbb{Q}$-lattices modulo the equivalence relation of commensurability is best described with the tools of noncommutative geometry, as explained in [10].

We will be concerned here only with the case of $n = 1$ or $n = 2$ and we will consider also the set of commensurability classes of $\mathbb{Q}$-lattices up to scaling, where the scaling action is given by the group $S = \mathbb{R}_+^*$ in the 1-dimensional case and $S = \mathbb{C}^*$ in the 2-dimensional case.

In these cases, one can first consider the groupoid $\mathcal{R}$ of the equivalence relation of commensurability on the set of $\mathbb{Q}$-lattices (not up to scaling). This is a locally compact étale groupoid. When considering the quotient by the scaling action, the algebra of coordinates associated to the quotient $\mathcal{R}/S$ is obtained by restricting the convolution product of the algebra of $\mathcal{R}$ to weight zero functions with S-compact support. The algebra obtained this way, which is unital in the 1-dimensional case, but not in the 2-dimensional case, has a natural time evolution given by the ratio of the covolumes of a pair of commensurable lattices. Every unit $y \in \mathcal{R}^{(0)}$ of $\mathcal{R}$ defines a representation π_y by left convolution of the algebra of $\mathcal{R}$ on the Hilbert space $\mathcal{H}_y = \ell^2(\mathcal{R}_y)$, where $\mathcal{R}_y$ is the set of elements with source y. This construction passes to the quotient by the scaling action of S. Representations corresponding to points that acquire a nontrivial automorphism group will no longer be irreducible. If the unit $y \in \mathcal{R}^{(0)}$ corresponds to an invertible $\mathbb{Q}$-lattice, then π_y is a positive energy representation.

In both the 1-dimensional and the 2-dimensional case, the set of extremal KMS states at low temperature is given by a classical adèlic quotient, namely, by the Shimura varieties for GL_1 and GL_2, respectively, hence we argue here that the noncommutative space describing commensurability classes of $\mathbb{Q}$-lattices up to scale can be thought of as a *noncommutative Shimura variety*, whose set of classical points is the corresponding classical Shimura variety.

In both cases, a crucial step for the arithmetic properties of the action of symmetries on extremal KMS states at zero temperature is the choice of an arithmetic subalgebra of the system, on which the extremal KMS_∞ states are evaluated. Such choice gives the underlying noncommutative space a more rigid structure, of "noncommutative arithmetic variety".

3.1 Tower Power

If V is an algebraic variety – or a scheme or a stack – over a field k, a "tower" $\mathcal{T}$ over V is a family V_i $(i \in \mathcal{I})$ of finite (possibly branched) covers of V such that for any $i, j \in \mathcal{I}$, there is a $l \in \mathcal{I}$ with V_l a cover of V_i and V_j. Thus, $\mathcal{I}$ is a partially ordered set. This gives a corresponding compatible system of covering maps $V_i \to V$. In case of a tower over a pointed variety (V, v), one fixes a point v_i over v in each V_i. Even though V_i may not be irreducible, we shall allow ourselves to loosely refer to V_i as a variety. It is convenient to view a "tower" $\mathcal{T}$ as a category $\mathcal{C}$ with objects $(V_i \to V)$ and morphisms $\mathrm{Hom}(V_i, V_j)$ being maps of covers of V. One has the group $\mathrm{Aut}_\mathcal{T}(V_i)$ of invertible self-maps of V_i over V (the group of deck transformations); the deck transformations are not required to preserve the points v_i. There is a (profinite) group of symmetries associated to a tower, namely

$$\mathcal{G} := \varprojlim_i \mathrm{Aut}_\mathcal{T}(V_i). \tag{14}$$

The simplest example of a tower is the "fundamental group" tower associated with a (smooth connected) complex algebraic variety (V, v) and its universal covering $(\tilde{V}, \tilde{v})$. Let $\mathcal{C}$ be the category of all finite étale (unbranched) intermediate covers $\tilde{V} \to W \to V$ of V. In this case, the symmetry group $\mathcal{G}$ of (14) is the algebraic fundamental group of V, which is also the profinite completion of the (topological) fundamental group $\pi_1(V, v)$. (For the theory of étale covers and fundamental groups, we refer the interested reader to SGA1.) Simple variants of this example include allowing controlled ramification. Other examples of towers are those defined by iteration of self maps of algebraic varieties.

For us, the most important examples of "towers" will be the cyclotomic tower and the modular tower[4]. Another very interesting case of towers is that of more general Shimura varieties. These, however, will not be treated in this paper. (For a systematic treatment of quantum statistical mechanical systems associated to general Shimura varieties see [13] and upcoming work by the same authors.)

3.2 The Cyclotomic Tower and the BC System

In the case of $\mathbb{Q}$, an explicit description of $\mathbb{Q}^{ab}$ is provided by the Kronecker–Weber theorem. This shows that the field $\mathbb{Q}^{ab}$ is equal to $\mathbb{Q}^{cyc}$, the field obtained by attaching all roots of unity to $\mathbb{Q}$. Namely, $\mathbb{Q}^{ab}$ is obtained by attaching the values of the exponential function $exp(2\pi i z)$ at the torsion points of the circle group $\mathbb{R}/\mathbb{Z}$. Using the isomorphism of abelian groups $\bar{\mathbb{Q}}^*_{tors} \cong \mathbb{Q}/\mathbb{Z}$ and the identification $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z}) = \mathrm{GL}_1(\hat{\mathbb{Z}}) = \hat{\mathbb{Z}}^*$, the restriction to $\bar{\mathbb{Q}}^*_{tors}$ of the natural action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\bar{\mathbb{Q}}^*$ factors as

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})^{ab} = \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \xrightarrow{\sim} \hat{\mathbb{Z}}^*.$$

Geometrically, the above setting can be understood in terms of the *cyclotomic tower*. This has base Spec $\mathbb{Z} = V_1$. The family is Spec $\mathbb{Z}[\zeta_n] = V_n$ where ζ_n is a primitive n-th root of unity ($n \in \mathbb{N}^*$). The set Hom $(V_m \to V_n)$, non-trivial for $n|m$, corresponds to the map $\mathbb{Z}[\zeta_n] \hookrightarrow \mathbb{Z}[\zeta_m]$ of rings. The group $\mathrm{Aut}(V_n) = \mathrm{GL}_1(\mathbb{Z}/n\mathbb{Z})$ is the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$. The group of symmetries (14) of the tower is then

$$\mathcal{G} = \varprojlim_n \mathrm{GL}_1(\mathbb{Z}/n\mathbb{Z}) = \mathrm{GL}_1(\hat{\mathbb{Z}}), \tag{15}$$

which is isomorphic to the Galois group $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ of the maximal abelian extension of $\mathbb{Q}$.

The classical object that we consider, associated to the cyclotomic tower, is the Shimura variety given by the adèlic quotient

$$Sh(\mathrm{GL}_1, \{\pm 1\}) = \mathrm{GL}_1(\mathbb{Q})\backslash(\mathrm{GL}_1(\mathbb{A}_f) \times \{\pm 1\}) = \mathbb{A}_f^*/\mathbb{Q}_+^*. \tag{16}$$

[4]In this paper, we reserve the terminology "modular tower" for the tower of modular curves. It would be interesting to investigate in a similar perspective the more general theory of modular towers in the sense of M.Fried.

Now we consider the space of 1-dimensional $\mathbb{Q}$-lattices up to scaling modulo commensurability. This can be described as follows ([10]).

In one dimension, every $\mathbb{Q}$-lattice is of the form

$$(\Lambda, \phi) = (\lambda\,\mathbb{Z}, \lambda\,\rho), \tag{17}$$

for some $\lambda > 0$ and some $\rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$. Since we can identify $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ endowed with the topology of pointwise convergence with

$$\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}\,, \tag{18}$$

we obtain that the algebra $C(\hat{\mathbb{Z}})$ is the algebra of coordinates of the space of 1-dimensional $\mathbb{Q}$-lattices up to scaling.

In fact, using the identification

$$\mathbb{Q}/\mathbb{Z} \cong \mathbb{A}_f/\hat{\mathbb{Z}}$$

one gets a natural character e of $\mathbb{A}_f/\hat{\mathbb{Z}}$ such that

$$e(r) = e^{2\pi i\, r} \quad \forall r \in \mathbb{Q}/\mathbb{Z}$$

and a pairing of $\mathbb{Q}/\mathbb{Z}$ with $\hat{\mathbb{Z}}$ such that

$$\langle r,\, x\rangle = e(r\,x) \quad \forall r \in \mathbb{Q}/\mathbb{Z}\,,\ x \in \hat{\mathbb{Z}}.$$

Thus, we identify the group $\hat{\mathbb{Z}}$ with the Pontrjagin dual of $\mathbb{Q}/\mathbb{Z}$ and we obtain a corresponding identification

$$C^*(\mathbb{Q}/\mathbb{Z}) \cong C(\hat{\mathbb{Z}}). \tag{19}$$

The group of deck transformations $\mathcal{G} = \hat{\mathbb{Z}}^*$ of the cyclotomic tower acts by automorphisms on the algebra of coordinates $C(\hat{\mathbb{Z}})$ in the obvious way. In addition to this action, there is a semigroup action of $\mathbb{N}^\times = \mathbb{Z}_{>0}$ implementing the commensurability relation. This is given by endomorphisms that move vertically across the levels of the cyclotomic tower. They are given by

$$\alpha_n(f)(\rho) = \begin{cases} f(n^{-1}\rho),\ \rho \in n\hat{\mathbb{Z}} \\ 0 \qquad\qquad \text{otherwise.} \end{cases} \tag{20}$$

Namely, α_n is the isomorphism of $C(\hat{\mathbb{Z}})$ with the reduced algebra $C(\hat{\mathbb{Z}})_{\pi_n}$ by the projection π_n given by the characteristic function of $n\hat{\mathbb{Z}} \subset \hat{\mathbb{Z}}$. Notice that the action (20) cannot be restricted to the set of invertible $\mathbb{Q}$-lattices, since the range of π_n is disjoint from them.

The algebra of coordinates $\mathcal{A}_1$ on the *noncommutative* space of equivalence classes of 1-dimensional $\mathbb{Q}$-lattices modulo scaling, with respect to the equivalence relation of commensurability, is given then by the semigroup crossed product

$$\mathcal{A} = C(\hat{\mathbb{Z}}) \rtimes_\alpha \mathbb{N}^\times . \tag{21}$$

Equivalently, we are considering the convolution algebra of the groupoid $\mathcal{R}_1/\mathbb{R}_+^*$ given by the quotient by scaling of the groupoid of the equivalence relation of commensurability on 1-dimensional $\mathbb{Q}$-lattices, namely, $\mathcal{R}_1/\mathbb{R}_+^*$ has as algebra of coordinates the functions $f(r,\rho)$, for $\rho \in \hat{\mathbb{Z}}$ and $r \in \mathbb{Q}^*$ such that $r\rho \in \hat{\mathbb{Z}}$, where $r\rho$ is the product in $\mathbb{A}_f$. The product in the algebra is given by the associative convolution product

$$f_1 * f_2\,(r,\rho) = \sum_{s : s\rho \in \hat{\mathbb{Z}}} f_1(rs^{-1}, s\rho) f_2(s,\rho), \tag{22}$$

and the adjoint is given by $f^*(r,\rho) = \overline{f(r^{-1}, r\rho)}$.

This is the C^*-algebra of the Bost–Connes (BC) system [3]. It has a natural time evolution σ_t determined by the ratio of the covolumes of two commensurable $\mathbb{Q}$-lattices,

$$\sigma_t(f)(r,\rho) = r^{it}\, f(r,\rho). \tag{23}$$

The algebra $\mathcal{A}_1$ was originally defined as a Hecke algebra for the almost normal pair of solvable groups $P_{\mathbb{Z}}^+ \subset P_{\mathbb{Q}}^+$, where P is the algebraic $ax+b$ group and P^+ is the restriction to $a > 0$ (*cf.* [3]).

As a set, the space of commensurability classes (*cf.* Definition 5) of 1-dimensional $\mathbb{Q}$-lattices up to scaling can also be described by the quotient

$$\mathrm{GL}_1(\mathbb{Q})\backslash \mathbb{A}^{\cdot}/\mathbb{R}_+^* = \mathrm{GL}_1(\mathbb{Q})\backslash(\mathbb{A}_f \times \{\pm 1\}) = \mathbb{A}_f/\mathbb{Q}_+^*, \tag{24}$$

where $\mathbb{A}^{\cdot} := \mathbb{A}_f \times \mathbb{R}^*$ is the set of adèles with nonzero archimedean component. Rather than considering this quotient set theoretically, we regard it as a noncommutative space, so as to be able to extend to it the ordinary tools of geometry that can be applied to the "good" quotient (16).

The noncommutative algebra of coordinates of (24) is the crossed product

$$C_0(\mathbb{A}_f) \rtimes \mathbb{Q}_+^*. \tag{25}$$

This is Morita equivalent to the algebra (21). In fact, (21) is obtained as a full corner of (25),

$$C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}^\times = \left(C_0(\mathbb{A}_f) \rtimes \mathbb{Q}_+^*\right)_\pi ,$$

by compression with the projection π given by the characteristic function of $\hat{\mathbb{Z}} \subset \mathbb{A}_f$ (*cf.* [19]).

The quotient (24) with its noncommutative algebra of coordinates (25) can then be thought of as the *noncommutative Shimura variety*

$$Sh^{(nc)}(\mathrm{GL}_1, \{\pm 1\}) := \mathrm{GL}_1(\mathbb{Q})\backslash(\mathbb{A}_f \times \{\pm 1\}) = \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}^{\cdot}/\mathbb{R}_+^*, \tag{26}$$

whose set of classical points is the well behaved quotient (16).

The noncommutative space of adèle classes, used in [8] in order to obtain the spectral realization of the zeros of the Riemann zeta function, is closely related to the noncommutative Shimura variety (26). First, one replaces $\mathbb{A}^{\cdot}$ with $\mathbb{A}$, as in [8]. This means adding the trivial lattice (the point $\lambda = 0$ in the $\mathbb{R}$-component of $\mathbb{A}$), with a possibly nontrivial $\mathbb{Q}$-structure (given by some $\rho \neq 0$ in $\mathbb{A}_f$). One obtains this way the space

$$\mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}/\mathbb{R}_+^*. \tag{27}$$

One then obtains the space of adèle classes of [8] as the dual to (27) under the duality given by taking the crossed product by the time evolution (*i.e.* in the duality between type II and type III factors of [6]). The space obtained this way is a principal $\mathbb{R}_+^*$-bundle over the noncommutative space

$$\mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}/\mathbb{R}_+^*. \tag{28}$$

3.3 Arithmetic Structure of the BC System

The results of [3] show that the Galois theory of the cyclotomic field $\mathbb{Q}^{cycl}$ appears naturally in the BC system when considering the action of the group of symmetries of the system on the extremal KMS states at zero temperature.

In the case of 1-dimensional $\mathbb{Q}$-lattices up to scaling, the algebra of coordinates $C(\hat{\mathbb{Z}})$ can be regarded as the algebra of *homogeneous functions of weight zero* on the space of 1-dimensional $\mathbb{Q}$-lattices. As such, there is a natural choice of an arithmetic subalgebra.

This is obtained in [10] by considering functions on the space of 1-dimensional $\mathbb{Q}$-lattices of the form

$$\epsilon_{1,a}(\Lambda, \phi) = \sum_{y \in \Lambda + \phi(a)} y^{-1}, \tag{29}$$

for any $a \in \mathbb{Q}/\mathbb{Z}$. This is well defined, for $\phi(a) \neq 0$, using the summation $\lim_{N\to\infty} \sum_{-N}^{N}$. One can then form the weight zero functions

$$e_{1,a} := c\, \epsilon_{1,a}, \tag{30}$$

where $c(\Lambda)$ is proportional to the covolume $|\Lambda|$ and normalized so that $(2\pi\sqrt{-1})c(\mathbb{Z}) = 1$. The rational subalgebra $\mathcal{A}_{1,\mathbb{Q}}$ of (21) is the $\mathbb{Q}$-algebra generated by the functions $e_{1,a}(r, \rho) := e_{1,a}(\rho)$ and by the functions $\mu_n(r, \rho) = 1$ for $r = n$ and zero otherwise. The latter implement the semigroup action of $\mathbb{N}^{\times}$ in (21).

As proved in [10], the algebra $\mathcal{A}_{1,\mathbb{Q}}$ is the same as the rational subalgebra considered in [3], generated over $\mathbb{Q}$ by the μ_n and the exponential functions

$$e(r)(\rho) := \exp(2\pi i \rho(r)), \quad \text{for } \rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}), \quad \text{and } r \in \mathbb{Q}/\mathbb{Z}, \tag{31}$$

with relations $e(r+s) = e(r)e(s)$, $e(0) = 1$, $e(r)^* = e(-r)$, $\mu_n^* \mu_n = 1$, $\mu_k \mu_n = \mu_{kn}$, and

$$\mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s). \tag{32}$$

The C^*-completion of $\mathcal{A}_{1,\mathbb{Q}} \otimes \mathbb{C}$ gives (21).

The algebra (21) has irreducible representations on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}^\times)$, parameterized by elements $\alpha \in \hat{\mathbb{Z}}^* = \mathrm{GL}_1(\hat{\mathbb{Z}})$. Any such element defines an embedding $\alpha : \mathbb{Q}^{cycl} \hookrightarrow \mathbb{C}$ and the corresponding representation is of the form

$$\pi_\alpha(e(r))\, \epsilon_k = \alpha(\zeta_r^k)\, \epsilon_k \qquad \pi_\alpha(\mu_n)\, \epsilon_k = \epsilon_{nk}. \tag{33}$$

The Hamiltonian implementing the time evolution σ_t on $\mathcal{H}$ is of the form $H\, \epsilon_k = \log k \ \epsilon_k$ and the partition function of the BC system is then the Riemann zeta function

$$Z(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right) = \sum_{k=1}^{\infty} k^{-\beta} = \zeta(\beta).$$

The set $\mathcal{E}_\beta$ of extremal KMS-states of the BC system enjoys the following properties (*cf.* [3]):

Theorem 6. (Bost–Connes [3]) *For the system* $(\mathcal{A}_1, \sigma_t)$ *described above, the structure of KMS states is the following.*

- $\mathcal{E}_\beta$ *is a singleton for all* $0 < \beta \leq 1$. *This unique KMS state takes values*

$$\varphi_\beta(e(m/n)) = f_{-\beta+1}(n)/f_1(n),$$

where

$$f_k(n) = \sum_{d|n} \mu(d)(n/d)^k,$$

with μ *the Möbius function, and* f_1 *is the Euler totient function.*
- *For* $1 < \beta \leq \infty$, *elements of* $\mathcal{E}_\beta$ *are indexed by the classes of invertible* $\mathbb{Q}$*-lattices* $\rho \in \hat{\mathbb{Z}}^* = \mathrm{GL}_1(\hat{\mathbb{Z}})$, *hence by the classical points* (16) *of the noncommutative Shimura variety* (26),

$$\mathcal{E}_\beta \cong \mathrm{GL}_1(\mathbb{Q}) \backslash \mathrm{GL}_1(\mathbb{A}) / \mathbb{R}_+^* \cong C_\mathbb{Q}/D_\mathbb{Q} \cong \mathbb{I}_\mathbb{Q}/\mathbb{Q}_+^*, \tag{34}$$

with $\mathbb{I}_\mathbb{Q}$ *as in* (1). *In this range of temperatures, the values of states* $\varphi_{\beta,\rho} \in \mathcal{E}_\beta$ *on the elements* $e(r) \in \mathcal{A}_{1,\mathbb{Q}}$ *is given, for* $1 < \beta < \infty$ *by polylogarithms evaluated at roots of unity, normalized by the Riemann zeta function,*

$$\varphi_{\beta,\rho}(e(r)) = \frac{1}{\zeta(\beta)} \sum_{n=1}^{\infty} n^{-\beta} \rho(\zeta_r^k).$$

- *The group* $\mathrm{GL}_1(\hat{\mathbb{Z}})$ *acts by automorphisms of the system. The induced action of* $GL_1(\hat{\mathbb{Z}})$ *on the set of extreme KMS states below critical temperature is free and transitive.*
- *The extreme KMS states at* $(\beta = \infty)$ *are* fabulous states *for the field* $K = \mathbb{Q}$*, namely* $\varphi(\mathcal{A}_{1,\mathbb{Q}}) \subset \mathbb{Q}^{cycl}$ *and the class field theory isomorphism* $\theta : \mathrm{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q}) \xrightarrow{\cong} \hat{\mathbb{Z}}^*$ *intertwines the Galois action on values with the action of* $\hat{\mathbb{Z}}^*$ *by symmetries,*

$$\gamma\, \varphi(x) = \varphi(\theta(\gamma)\, x), \tag{35}$$

for all $\varphi \in \mathcal{E}_\infty$*, for all* $\gamma \in \mathrm{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q})$ *and for all* $x \in \mathcal{A}_{1,\mathbb{Q}}$*.*

This shows that the BC system is a solution to Problem 1 for $K = \mathbb{Q}$.

We return now to the discussion of Section 2.3 about the comparison between the problem of fabulous states (Problem 4) and explicit class field theory. In the case of the BC system, the "trivial system" mentioned in Section 2.3 corresponds to considering the abelian part of the algebra restricted to invertible $\mathbb{Q}$-lattices. On this the restriction of the time evolution is trivial. The simple transitivity of the action of the Galois group on S shows trivially that this restriction $\mathcal{B}$ is isomorphic to $\mathbb{Q}^{ab}$, viewed as a subalgebra of $C(S)$.

It is important to understand, already in the case of the BC system, how the "rational subalgebra" $\mathcal{A}_{1,\mathbb{Q}}$ is simpler than the restriction $\mathcal{B}$ of its abelian part to the space $S \cong \mathbb{A}_f^*/\mathbb{Q}_+^*$ of invertible $\mathbb{Q}$-lattices. This depends essentially on the fact of considering all $\mathbb{Q}$-lattices, not just the invertible ones.

First one can see easily that the "trivial system" $(\mathcal{B}, \sigma_t)$ obtained this way is no longer a solution to Problem 1, as it does not fulfill any of the deeper properties regarding phase transitions and the partition function. However, the main point we wish to discuss here is the fact that, passing from $\mathcal{B} \cong \mathbb{Q}^{ab}$ to the algebra $\mathcal{A}_{1,\mathbb{Q}}$ gives us a *simpler presentation.*

The presentation of $\mathbb{Q}^{ab}$ is given by the cyclotomic theory, while our rational algebra is simply the group ring of $\mathbb{Q}/\mathbb{Z}$ with coefficients in $\mathbb{Q}$. We can use this to obtain a presentation of $\mathbb{Q}^{ab}$ which does not use any direct reference to field extensions in the following way.

Let $H = \mathbb{Q}/\mathbb{Z}$ be the additive group given by the quotient of $\mathbb{Q}$ by $\mathbb{Z} \subset \mathbb{Q}$. Consider the inclusion of algebras

$$\mathbb{Q}[H] \subset C^*(H) \tag{36}$$

and let $J \subset C^*(H)$ be the ideal generated by the idempotents

$$\pi_m = \frac{1}{m} \sum_{r \in H\,,\; m\, r = 0} u(r), \tag{37}$$

where $m > 1$ and the $u(r)$ form the canonical basis of $\mathbb{Q}[H]$.

Proposition 7. *With H and J as above, the following holds.*

1. *The quotient*
$$\mathbb{Q}[H]/(\mathbb{Q}[H] \cap J) \tag{38}$$
is a field which is isomorphic to $\mathbb{Q}^{cycl} \cong \mathbb{Q}^{ab}$.
2. *$\mathbb{Q}[H] \cap J$ is equal to the ideal generated by the π_m in $\mathbb{Q}[H]$.*

Proof. 1) We use the identification of the group $\hat{\mathbb{Z}}$ with the Pontrjagin dual of H and the isomorphism of algebras (19).

Under this identification, the ideal J corresponds to the ideal of functions that vanish on the closed subset
$$\hat{\mathbb{Z}}^* \subset \hat{\mathbb{Z}}.$$
The quotient map $C^*(H) \to C^*(H)/J$ is just the restriction map
$$f \in C(\hat{\mathbb{Z}}) \mapsto f|_{\hat{\mathbb{Z}}^*}\,. \tag{39}$$

In fact, first notice that the restriction of π_m to $\hat{\mathbb{Z}}^*$ vanishes since the sum of all roots of unit of order m is zero. This shows that J is contained in the kernel of the restriction map (39). Moreover, a character of the C^*-algebra $C^*(H)/J$ is given by a point $a \in \hat{\mathbb{Z}}$ on which all elements of J vanish. However, if $a \notin \hat{\mathbb{Z}}^*$, then there exists $m > 1$ such that $a \in m \times \hat{\mathbb{Z}}$ and one gets $\pi_m(a) = 1 \neq 0$.

The multiplicative group $G = \hat{\mathbb{Z}}^*$ acts by multiplication on both $\mathbb{A}_f/\hat{\mathbb{Z}} \cong H$ and on $\hat{\mathbb{Z}}$ and these actions are compatible with the pairing
$$\langle g\, r,\, x\rangle = \langle r,\, g\, x\rangle,$$
and therefore with the isomorphism (19), and with the actions of G by automorphisms
$$G \subset \mathrm{Aut}\; C^*(H)\,, \quad G \subset \mathrm{Aut}\; C(\hat{\mathbb{Z}}).$$

The group G is also identified with the Galois group of the cyclotomic field, *i.e.* the subfield $\mathbb{Q}^{cyc} \subset \mathbb{C}$ generated by all the roots of unity of all orders. For any element $v \in \hat{\mathbb{Z}}^* \subset \hat{\mathbb{Z}}$, let φ_v denote the evaluation map $\varphi_v : C(\hat{\mathbb{Z}}) \to \mathbb{C}$, with $\varphi_v(f) = f(v)$. These maps fulfill the condition
$$g(\varphi_v(f)) = \varphi_v(g(f)) = \varphi_{gv}(f). \tag{40}$$
for all $g \in G$ and for all $f \in \mathbb{Q}[H]$, where we use the inclusion (36) and the identification (19).

Thus, the restriction of $f \in \mathbb{Q}[H]$ to $\hat{\mathbb{Z}}^* \subset \hat{\mathbb{Z}}$ is determined by the single field element $\varphi_1(f) \in \mathbb{Q}^{cyc}$. The restriction map (39) then induces an isomorphism
$$\mathbb{Q}[H]/(\mathbb{Q}[H] \cap J) \xrightarrow{\cong} \mathbb{Q}^{cyc}\,. \tag{41}$$

2) Let J_0 be the ideal generated by the π_m in $\mathbb{Q}[H]$. It is enough to show that, for Φ_n the n-th cyclotomic polynomial (*cf.* [26]), one has $\Phi_n(u(1/n)) \in J_0$. When n is prime one has $\Phi_n(u(1/n)) = n\,\pi_n$. In general, if we write

$$\sigma_k(x) = \sum_0^{k-1} x^j, \tag{42}$$

then we obtain that Φ_n is the g.c.d. of the polynomials $\sigma_m(x^d)$, where d divides n and $m = n/d$. For $x = u(1/n)$, one has $\sigma_m(x^d) = m\,\pi_m \in J_0$, for any divisor $d|n$. Thus, we obtain $\Phi_n(u(1/n)) \in J_0$ as required. ◻

Recall that $\pi_m = \mu_m\,\mu_m^*$ and $\mu_m^*\,\mu_m = 1$. Thus, the KMS_∞ states of the BC system vanish identically on the π_m. The KMS_β condition for $1 < \beta < \infty$ shows that KMS_β states take value $m^{-\beta}$ on π_m. It follows that the restriction of KMS_∞ states to the abelian part vanish identically on J.

In our setup we proved in [10] that the rational subalgebra $\mathcal{A}_{1,\mathbb{Q}}$ of the BC system is generated by the natural simple functions of $\mathbb{Q}$-lattices given by summation and is actually the same as $\mathbb{Q}[H] \subset C^*(H)$. (This follows using the isomorphism of $C^*(H)$ with continuous functions of weight zero on $\mathbb{Q}$-lattices.) Thus, one can reasonably expect that similar simplifications will occur in the case of imaginary quadratic fields.

In [10] we described a set of non-trivial relations for the algebra $\mathcal{A}_{2,\mathbb{Q}}$ of the GL_2-system (which will be recalled in Definition 11 below). One can consider the specialization of these relations to the value of the j-invariant of an elliptic curve with complex multiplication will provide relations for the CM system (described in Section 4 below). We expect that this will provide the key for an explicit presentation of the rational subalgebra $\mathcal{A}_{K,\mathbb{Q}}$ of the CM system as an algebra over the Hilbert class field $K(j)$.

3.4 The Modular Tower and the GL$_2$-System

Modular curves arise as moduli spaces of elliptic curves endowed with additional level structure. Every congruence subgroup Γ' of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ defines a modular curve $Y_{\Gamma'}$; we denote by $X_{\Gamma'}$ the smooth compactification of the affine curve $Y_{\Gamma'}$ obtained by adding cusp points. Especially important among these are the modular curves $Y(n)$ and $X(n)$ corresponding to the principal congruence subgroups $\Gamma(n)$ for $n \in \mathbb{N}^*$. Any $X_{\Gamma'}$ is dominated by an $X(n)$. We refer to [17, 32] for more details. We have the following descriptions of the modular tower.

Compact version: The base is $V = \mathbb{P}^1$ over $\mathbb{Q}$. The family is given by the modular curves $X(n)$, considered over the cyclotomic field $\mathbb{Q}(\zeta_n)$ [25]. We note that $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})/\pm 1$ is the group of automorphisms of the projection $V_n = X(n) \to X(1) = V_1 = V$. Thus, we have

$$\mathcal{G} = \mathrm{GL}_2(\hat{\mathbb{Z}})/\{\pm 1\} = \varprojlim_n \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})/\{\pm 1\}. \tag{43}$$

Non-compact version: The open modular curves $Y(n)$ form a tower with base the j-line Spec $\mathbb{Q}[j] = \mathbb{A}^1 = V_1 - \{\infty\}$. The ring of modular functions is the union of the rings of functions of the $Y(n)$, with coefficients in $\mathbb{Q}(\zeta_n)$ [17].

This shows how the modular tower is a natural geometric way of passing from $\mathrm{GL}_1(\hat{\mathbb{Z}})$ to $\mathrm{GL}_2(\hat{\mathbb{Z}})$. The formulation that is most convenient in our setting is the one given in terms of Shimura varieties. In fact, rather than the modular tower defined by the projective limit

$$Y = \varprojlim_n Y(n) \tag{44}$$

of the modular curves $Y(n)$, it is better for our purposes to consider the Shimura variety

$$Sh(\mathbb{H}^{\pm}, \mathrm{GL}_2) = \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}) = \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/\mathbb{C}^*, \tag{45}$$

of which (44) is a connected component. Here $\mathbb{H}^{\pm}$ denotes the upper and lower half planes.

It is well known that, for arithmetic purposes, it is always better to work with nonconnected rather than with connected Shimura varieties (*cf. e.g.* [25]). The simple reason why it is necessary to pass to the nonconnected case is the following. The varieties in the tower are arithmetic varieties defined over number fields. However, the number field typically changes along the levels of the tower ($Y(n)$ is defined over the cyclotomic field $\mathbb{Q}(\zeta_n)$). Passing to nonconnected Shimura varieties allows precisely for the definition of a canonical model where the whole tower is defined over the same number field.

This distinction is important to our viewpoint, since we want to work with noncommutative spaces endowed with an arithmetic structure, specified by the choice of an arithmetic subalgebra.

Every 2-dimensional $\mathbb{Q}$-lattice can be described by data

$$(\Lambda, \phi) = (\lambda(\mathbb{Z} + \mathbb{Z}z), \lambda\alpha), \tag{46}$$

for some $\lambda \in \mathbb{C}^*$, some $z \in \mathbb{H}$, and $\alpha \in M_2(\hat{\mathbb{Z}})$ (using the basis $(1, -z)$ of $\mathbb{Z}+\mathbb{Z}z$ as in (87) [10] to view α as a map ϕ). The diagonal action of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ yields isomorphic $\mathbb{Q}$-lattices, and (*cf.* (87) [10]) the space of 2-dimensional $\mathbb{Q}$-lattices up to scaling can be identified with the quotient

$$\Gamma\backslash(\mathrm{M}_2(\hat{\mathbb{Z}}) \times \mathbb{H}). \tag{47}$$

The relation of commensurability is implemented by the partially defined action of $\mathrm{GL}_2^+(\mathbb{Q})$ on (47).

We denote by $\mathcal{R}_2$ the groupoid of the commensurability relation on 2-dimensional $\mathbb{Q}$-lattices not up to scaling.

The groupoid $\mathcal{R}_2$ has, as (noncommutative) algebra of coordinates, the convolution algebra of $\Gamma \times \Gamma$-invariant functions on

$$\tilde{\mathcal{U}} = \{(g, \alpha, u) \in \mathrm{GL}_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathrm{GL}_2^+(\mathbb{R}) \,|\, g\alpha \in M_2(\hat{\mathbb{Z}})\}. \tag{48}$$

Up to Morita equivalence, this can also be described as the crossed product

$$C_0(M_2(\mathbb{A}_f) \times \mathrm{GL}_2(\mathbb{R})) \rtimes \mathrm{GL}_2(\mathbb{Q}). \tag{49}$$

Passing to $\mathbb{Q}$-lattices up to scaling corresponds to taking the quotient

$$Z := \mathcal{R}_2/\mathbb{C}^*. \tag{50}$$

The action of $\mathbb{C}^*$ on $\mathbb{Q}$-lattices preserves the commensurability relation. However, the action is not free due to the presence of lattices $L = (0, z)$, where $z \in \Gamma\backslash\mathbb{H}$ has nontrivial automorphisms. Thus, the following problem arises.

Claim. The quotient $Z = \mathcal{R}_2/\mathbb{C}^*$ is no longer a groupoid.

Proof. This can be seen in the following simple example. Consider the two $\mathbb{Q}$-lattices $(\alpha_1, z_1) = (0, 2\sqrt{-1})$ and $(\alpha_2, z_2) = (0, \sqrt{-1})$. The composite

$$((\alpha_1, z_1), (\alpha_2, z_2)) \circ ((\alpha_2, z_2), (\alpha_1, z_1))$$

is equal to the identity $((\alpha_1, z_1), (\alpha_1, z_1))$. We can also consider the composition

$$(\sqrt{-1}\,(\alpha_1, z_1), \sqrt{-1}\,(\alpha_2, z_2)) \circ ((\alpha_2, z_2), (\alpha_1, z_1)),$$

where $\sqrt{-1}\,(\alpha_2, z_2) = (\alpha_2, z_2)$, but this is not the identity, since $\sqrt{-1}\,(\alpha_1, z_1) \neq (\alpha_1, z_1)$. □

However, we can still define a convolution algebra for the quotient Z of (50), by restricting the convolution product of $\mathcal{R}_2$ to homogeneous functions of weight zero with $\mathbb{C}^*$-compact support, where a function f has weight k if it satisfies

$$f(g, \alpha, u\lambda) = \lambda^k f(g, \alpha, u), \quad \forall \lambda \in \mathbb{C}^*.$$

This is the analog of the description (21) for the 1-dimensional case.

Definition 8. *The noncommutative algebra of coordinates $\mathcal{A}_2$ is the Hecke algebra of functions on*

$$\mathcal{U} = \{(g, \alpha, z) \in \mathrm{GL}_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathbb{H},\ g\alpha \in M_2(\hat{\mathbb{Z}})\} \tag{51}$$

invariant under the $\Gamma \times \Gamma$ action

$$(g, \alpha, z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 \alpha, \gamma_2(z)), \tag{52}$$

with convolution

$$(f_1 * f_2)\,(g, \alpha, z) = \sum_{s \in \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q}),\, s\alpha \in M_2(\hat{\mathbb{Z}})} f_1(gs^{-1}, s\alpha, s(z))\, f_2(s, \alpha, z) \tag{53}$$

and adjoint $f^*(g, \alpha, z) = \overline{f(g^{-1}, g\alpha, g(z))}$.

This contains the classical Hecke operators (*cf.* (128) [10]). The time evolution determined by the ratio of covolumes of pairs of commensurable $\mathbb{Q}$-lattices is given by

$$\sigma_t(f)(g,\alpha,\tau) = \det(g)^{it} f(g,\alpha,\tau), \tag{54}$$

where, for the pair of commensurable $\mathbb{Q}$-lattices associated to (g,α,τ), one has

$$\det(g) = \text{covolume}(\Lambda')/\text{covolume}(\Lambda). \tag{55}$$

3.5 Tate Modules and Shimura Varieties

We now give a description closer to (25), which shows that again we can interpret the space of commensurability classes of 2-dimensional $\mathbb{Q}$-lattices up to scaling as a noncommutative version of the Shimura variety (45). More precisely, we give a reinterpretation of the notion of 2-dimensional $\mathbb{Q}$-lattices and commensurability, which may be useful in the context of similar quantum statistical mechanical systems associated to more general Shimura varieties as in [13].

Implicit in what follows is an isomorphism between $\mathbb{Q}/\mathbb{Z}$ and the roots of unity in $\mathbb{C}$. For instance, this can be given by the exponential function $e^{2\pi iz}$.

Proposition 9. *The data of a 2-dimensional $\mathbb{Q}$-lattice up to scaling are equivalent to the data of an elliptic curve E, together with a pair of points $\xi = (\xi_1, \xi_2)$ in the cohomology $H^1(E, \hat{\mathbb{Z}})$. Commensurability of 2-dimensional $\mathbb{Q}$-lattices up to scale is then implemented by an isogeny of the corresponding elliptic curves, with the elements ξ and ξ' related via the induced map in cohomology.*

Proof. The subgroup $\mathbb{Q}\Lambda/\Lambda$ of $\mathbb{C}/\Lambda = E$ is the torsion subgroup E_{tor} of the elliptic curve E. Thus, one can rewrite the map ϕ as a map $\mathbb{Q}^2/\mathbb{Z}^2 \to E_{tor}$. Using the canonical isomorphism $E[n] \xrightarrow{\sim} H^1(E, \mathbb{Z}/n\mathbb{Z})$, for $E[n] = \Lambda/n\Lambda$ the n-torsion points of E, one can interpret ϕ as a map $\mathbb{Q}^2/\mathbb{Z}^2 \to H^1(E, \mathbb{Q}/\mathbb{Z})$.

By taking $\text{Hom}(\mathbb{Q}/\mathbb{Z}, -)$, the map ϕ corresponds to a $\hat{\mathbb{Z}}$-linear map

$$\hat{\mathbb{Z}} \oplus \hat{\mathbb{Z}} \to H^1(E, \hat{\mathbb{Z}}) \tag{56}$$

or to a choice of two elements of the latter. In fact, we use here the identification

$$H^1(E, \hat{\mathbb{Z}}) \cong TE$$

of $H^1(E, \hat{\mathbb{Z}}) = H^1(E, \mathbb{Z}) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ with the total Tate module

$$\Lambda \otimes \hat{\mathbb{Z}} = \varprojlim_n E[n] = TE. \tag{57}$$

Thus, (56) gives a cohomological formulation of the $\hat{\mathbb{Z}}$-linear map $\phi : \hat{\mathbb{Z}} \oplus \hat{\mathbb{Z}} \to \Lambda \otimes \hat{\mathbb{Z}}$. Commensurability of $\mathbb{Q}$-lattices up to scale is rephrased as the condition that the elliptic curves are isogenous and the points in the Tate module are related via the induced map in cohomology. □

Another reformulation uses the Pontrjagin duality between profinite abelian groups and discrete torsion abelian groups given by $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$. This reformulates the datum ϕ of a $\mathbb{Q}$-lattice as a $\hat{\mathbb{Z}}$-linear map $\mathrm{Hom}(\mathbb{Q}\Lambda/\Lambda, \mathbb{Q}/\mathbb{Z}) \to \hat{\mathbb{Z}} \oplus \hat{\mathbb{Z}}$, which is identified with $\Lambda \otimes \hat{\mathbb{Z}} \to \hat{\mathbb{Z}} \oplus \hat{\mathbb{Z}}$. Here we use the fact that Λ and $\Lambda \otimes \hat{\mathbb{Z}} \cong H^1(E, \hat{\mathbb{Z}})$ are both self-dual (Poincaré duality of E). In this dual formulation commensurability means that the two maps agree on the intersection of the two commensurable lattices, $(\Lambda_1 \cap \Lambda_2) \otimes \hat{\mathbb{Z}}$.

With the formulation of Proposition 9, we can then give a new interpretation of the result of Proposition 43 of [10], which shows that the space of commensurability classes of 2-dimensional $\mathbb{Q}$-lattices up to scaling is described by the quotient

$$Sh^{(nc)}(\mathbb{H}^{\pm}, \mathrm{GL}_2) := \mathrm{GL}_2(\mathbb{Q}) \backslash (M_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}). \tag{58}$$

In fact, commensurability classes of $\mathbb{Q}$-lattices (Λ, ϕ) in $\mathbb{C}$ are the same as isogeny classes of data (E, η) of elliptic curves $E = \mathbb{C}/\Lambda$ and $\mathbb{A}_f$-homomorphisms

$$\eta : \mathbb{Q}^2 \otimes \mathbb{A}_f \to \Lambda \otimes \mathbb{A}_f, \tag{59}$$

with $\Lambda \otimes \mathbb{A}_f = (\Lambda \otimes \hat{\mathbb{Z}}) \otimes \mathbb{Q}$, where we can identify $\Lambda \otimes \hat{\mathbb{Z}}$ with the total Tate module of E, as in (57). Since the $\mathbb{Q}$-lattice need not be invertible, we do not require that η be an $\mathbb{A}_f$-isomorphism (*cf.* [25]).

In fact, the commensurability relation between $\mathbb{Q}$-lattices corresponds to the equivalence $(E, \eta) \sim (E', \eta')$ given by an isogeny $g : E \to E'$ and $\eta' = (g \otimes 1) \circ \eta$. Namely, the equivalence classes can be identified with the quotient of $M_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}$ by the action of $\mathrm{GL}_2(\mathbb{Q})$, $(\rho, z) \mapsto (g\rho, g(z))$.

Thus, (58) describes a noncommutative Shimura variety which has the Shimura variety (45) as the set of its classical points. The results of [10] show that, as in the case of the BC system, the set of low temperature extremal KMS states is a classical Shimura variety. We shall return to this in the next section.

Remark 10. Under the Gelfand–Naimark correspondence, unital C^*-algebras correspond to compact spaces. Thus, one can say that a noncommutative space X described by a C^*-algebra $\mathcal{A}$ is "Morita–compact" if the algebra $\mathcal{A}$ is Morita equivalent to a unital C^*-algebra.

This provides us with a notion of "Morita–compactification" for noncommutative spaces. In the case of the GL_2-system, we can replace (58) by the noncommutative space

$$\overline{Sh^{(nc)}}(\mathbb{H}^{\pm}, \mathrm{GL}_2) := \mathrm{GL}_2(\mathbb{Q}) \backslash (M_2(\mathbb{A}_f) \times \mathbb{P}^1(\mathbb{C})) \sim \mathrm{GL}_2(\mathbb{Q}) \backslash M_2(\mathbb{A})^{\cdot} / \mathbb{C}^*, \tag{60}$$

where we use the notation $M_2(\mathbb{A})^{\cdot} = M_2(\mathbb{A}_f) \times (M_2(\mathbb{R}) \setminus \{0\})$. This corresponds to allowing degenerations of the underlying lattice in $\mathbb{C}$ to a pseudolattice (*cf.* [22]), while maintaining the $\mathbb{Q}$-structure (*cf.* [10]). Thus, the space (60) can be thought of as obtained from the classical Shimura variety $Sh(\mathrm{GL}_2, \mathbb{H}^{\pm})$

by allowing all possible degenerations of the lattice, both at the archimedean and at the non-archimedean components. In the archimedan case, this accounts for adding the rank one matrices to $GL_2(\mathbb{R})$, so as to obtain $M_2(\mathbb{R}) \smallsetminus \{0\}/\mathbb{C}^* = \mathbb{P}^1(\mathbb{R})$, which corresponds to the pseudo-lattices. In the non-archimedean part one allows the presence of non-invertible $\mathbb{Q}$-lattices, which, by Proposition 9, can also be viewed (adelically) as an analogous degenerations (the two points ξ_i in the Tate module need not be independent). The "invertible part"

$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{P}^1(\mathbb{R})) \tag{61}$$

of the "boundary" gives the noncommutative modular tower considered in [9], [24], and [29], so that the full space (60) appears to be the most natural candidate for the geometry underlying the construction of a quantum statistical mechanical system adapted to the case of both imaginary and real quadratic fields (*cf.* [22] [23]).

3.6 Arithmetic Properties of the GL_2-System

The result proved in [10] for the GL_2-system shows that the action of symmetries on the extremal KMS states at zero temperature is now related to the Galois theory of the field of modular functions.

Since the arithmetic subalgebra for the BC system was obtained by considering weight zero lattice functions of the form (30), it is natural to expect that the analog for the GL_2-system will involve lattice functions given by the Eisenstein series, suitably normalized to weight zero, according to the analogy developed by Kronecker between trigonometric and elliptic functions, as outlined by A.Weil in [34]. This suggests that modular functions should appear naturally in the arithmetic subalgebra $\mathcal{A}_{2,\mathbb{Q}}$ of the GL_2-system, but that requires working with unbounded multipliers.

This is indeed the case for the arithmetic subalgebra $\mathcal{A}_{2,\mathbb{Q}}$ defined in [10], which we now recall. The main point of the following definition is that the rationl subalgebra of the GL_2-system is obtained by assigning the simplest condition of algebraicity on the parameters the algebra depends on.

The q-expansion at a cusp of modular functions of level N in the modular field F (*cf.* Section 1.1) has coefficients in the cyclotomic field $\mathbb{Q}(\zeta_N)$. Thus, the Galois group $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ acts on these coefficients through $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$. This action of $\hat{\mathbb{Z}}^* \simeq \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ on the coefficients of the q-expansion determines a homomorphism

$$\mathrm{cycl} : \hat{\mathbb{Z}}^* \to \mathrm{Aut}(F). \tag{62}$$

For $\rho \in \hat{\mathbb{Z}}^*$, this sends a modular function $f \in F$ with coefficients a_n of the q-expansion to the modular function $\mathrm{cycl}(\rho)(f)$ that has the $\rho(a_n)$ as coefficients.

If f is a continuous function on the quotient Z of (50), we write

$$f_{(g,\alpha)}(z) = f(g, \alpha, z)$$

so that $f_{(g,\alpha)} \in C(\mathbb{H})$. Here the coordinates (g, α, z) are as in (51), with $z \in \mathrm{GL}_2^+(\mathbb{R})/\mathbb{C}^*$.

For $p_N : M_2(\hat{\mathbb{Z}}) \to M_2(\mathbb{Z}/N\mathbb{Z})$ the canonical projection, we say that f is of level N if

$$f_{(g,\alpha)} = f_{(g,p_N(\alpha))} \quad \forall (g, \alpha).$$

Then f is completely determined by the functions

$$f_{(g,m)} \in C(\mathbb{H}), \qquad \text{for } m \in M_2(\mathbb{Z}/N\mathbb{Z}).$$

Definition 11. *[10] The arithmetic algebra $\mathcal{A}_{2,\mathbb{Q}}$ is a subalgebra of continuous functions on the quotient Z of* (50)*, with the convolution product* (53) *and with the properties:*

- *The support of f in $\Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q})$ is finite.*
- *The function f is of finite level.*
- *For all (g, m), the function f satisfies $f_{(g,m)} \in F$.*
- *The function f satisfies the* cyclotomic condition*:*

$$f_{(g,\alpha(u)m)} = \mathrm{cycl}(u)\, f_{(g,m)},$$

for all $g \in \mathrm{GL}_2^+(\mathbb{Q})$ diagonal and all $u \in \hat{\mathbb{Z}}^*$, with

$$\alpha(u) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$$

and cycl *as in* (62)*.*

Notice that the invariance $f(g\gamma, \alpha, z) = f(g, \gamma\alpha, \gamma(z))$, for all $\gamma \in \Gamma$ and for all $(g, \alpha, z) \in \mathcal{U}$, implies that $f_{(g,m)|\gamma} = f_{(g,m)}$, for all $\gamma \in \Gamma(N) \cap g^{-1}\Gamma g$, *i.e.* f is invariant under a congruence subgroup.

The cyclotomic condition is a consistency condition on the roots of unity that appear in the coefficients of the q-series. It only depends on the simple action of $\hat{\mathbb{Z}}^*$ on the coefficients, but it is crucial for the existence of "fabulous states" for the action of the Galois group of the modular field (*cf.* [10]).

It is important to remark here, for the later application to the case of imaginary quadratic fields, that *the Galois theory of the modular field is not built into the definition of the arithmetic algebra $\mathcal{A}_{2,\mathbb{Q}}$*. In fact, the properties we assume in Definition 11 arise by just requiring a natural algebraicity condition on the only continuous modulus q present in the algebra (while eliminating the trivial cases through the cyclotomic condition). The refined theory of automorphisms of the modular field does not enter in any way in the definition of the algebra.

For $\alpha \in M_2(\hat{\mathbb{Z}})$, let $G_\alpha \subset \mathrm{GL}_2^+(\mathbb{Q})$ be the set of

$$G_\alpha = \{g \in \mathrm{GL}_2^+(\mathbb{Q}) : \; g\alpha \in M_2(\hat{\mathbb{Z}})\}.$$

Then, as shown in [10], an element $y = (\alpha, z) \in M_2(\hat{\mathbb{Z}}) \times \mathbb{H}$ determines a unitary representation of the Hecke algebra $\mathcal{A}$ on the Hilbert space $\ell^2(\Gamma \backslash G_\alpha)$,

$$((\pi_y f)\xi)(g) := \sum_{s \in \Gamma \backslash G_\alpha} f(gs^{-1}, s\alpha, s(z))\, \xi(s), \quad \forall g \in G_\alpha \tag{63}$$

for $f \in \mathcal{A}$ and $\xi \in \ell^2(\Gamma \backslash G_\alpha)$.

Invertible $\mathbb{Q}$-lattices determine positive energy representations, due to the fact that the condition $g\alpha \in M_2(\hat{\mathbb{Z}})$ for $g \in \mathrm{GL}_2^+(\mathbb{Q})$ and $\alpha \in \mathrm{GL}_2(\hat{\mathbb{Z}})$ (invertible case) implies $g \in M_2(\mathbb{Z})^+$, hence the time evolution (54) is implemented by the positive Hamiltonian with spectrum $\{\log \det(m)\} \subset [0, \infty)$ for $m \in \Gamma \backslash M_2(\mathbb{Z})^+$. The partition function of the GL_2-system is then $Z(\beta) = \zeta(\beta)\zeta(\beta - 1)$. This shows that one can expect the system to have two phase transitions, which is in fact the case.

While the group $\mathrm{GL}_2(\hat{\mathbb{Z}})$ acts by automorphisms of the algebra of coordinates of $\mathcal{R}_2/\mathbb{C}^*$, *i.e.* the algebra of the quotient (58), the action of $\mathrm{GL}_2(\mathbb{A}_f)$ on the Hecke algebra $\mathcal{A}_2$ of coordinates of $\mathcal{R}_2/\mathbb{C}^*$ is by *endomorphisms*. More precisely, the group $\mathrm{GL}_2(\hat{\mathbb{Z}})$ of deck transformations of the modular tower still acts by automorphisms on this algebra, while $\mathrm{GL}_2^+(\mathbb{Q})$ acts by endomorphisms of the C^*-dynamical system, with the diagonal $\mathbb{Q}^*$ acting by inner, as in (7).

The group of symmetries $\mathrm{GL}_2(\mathbb{A}_f)$ preserves the arithmetic subalgebra; we remark that $\mathrm{GL}_2(\mathbb{A}_f) = \mathrm{GL}_2^+(\mathbb{Q}).\mathrm{GL}_2(\hat{\mathbb{Z}})$. In fact, the group $\mathbb{Q}^* \backslash \mathrm{GL}_2(\mathbb{A}_f)$ has an important arithmetic meaning: a result of Shimura (*cf.* [32], [21]) characterizes the automorphisms of the modular field by the exact sequence

$$0 \to \mathbb{Q}^* \to \mathrm{GL}_2(\mathbb{A}_f) \to \mathrm{Aut}(F) \to 0. \tag{64}$$

There is an induced action of $\mathbb{Q}^* \backslash \mathrm{GL}_2(\mathbb{A}_f)$ (symmetries modulo inner) on the KMS_β states of $(\mathcal{A}, \sigma_t)$, for $\beta < \infty$. The action of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ extends to KMS_∞ states, while the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on Σ_∞ is defined by the action at finite (large) β, by first "warming up" and then "cooling down" as in (9) (*cf.* [10]).

The result of [10] on the structure of KMS states for the GL_2 system is as follows.

Theorem 12. (Connes–Marcolli [10]) *For the system $(\mathcal{A}_2, \sigma_t)$ described above, the structure of the KMS states is as follows.*

- *There is no KMS state in the range $0 < \beta \leq 1$.*
- *In the range $\beta > 2$ the set of extremal KMS states is given by the invertible $\mathbb{Q}$-lattices, namely by the Shimura variety $Sh(\mathrm{GL}_2, \mathbb{H}^\pm)$,*

$$\mathcal{E}_\beta \cong \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}) / \mathbb{C}^*. \tag{65}$$

The explicit expression for these extremal KMS_β states is

$$\varphi_{\beta, L}(f) = \frac{1}{Z(\beta)} \sum_{m \in \Gamma \backslash M_2^+(\mathbb{Z})} f(1, m\alpha, m(z))\, \det(m)^{-\beta} \tag{66}$$

where $L = (\alpha, z)$ is an invertible $\mathbb{Q}$-lattice.

- *At* $\beta = \infty$, *and for* generic $L = (\alpha, \tau)$ *invertible (where generic means* $j(\tau) \notin \bar{\mathbb{Q}}$*), the values of the state* $\varphi_{\infty,L} \in \mathcal{E}_\infty$ *on elements of* $\mathcal{A}_{2,\mathbb{Q}}$ *lie in an embedded image in* $\mathbb{C}$ *of the modular field,*

$$\varphi(\mathcal{A}_{2,\mathbb{Q}}) \subset F_\tau, \tag{67}$$

and there is an isomorphism

$$\theta_\varphi : \mathrm{Aut}_{\mathbb{Q}}(F_\tau) \xrightarrow{\simeq} \mathbb{Q}^* \backslash \mathrm{GL}_2(\mathbb{A}_f), \tag{68}$$

depending on $L = (\alpha, \tau)$*, which intertwines the Galois action on the values of the state with the action of symmetries,*

$$\gamma\, \varphi(f) = \varphi(\theta_\varphi(\gamma) f), \qquad \forall f \in \mathcal{A}_{2,\mathbb{Q}}, \quad \forall \gamma \in \mathrm{Aut}_{\mathbb{Q}}(F_\tau). \tag{69}$$

3.7 Crossed Products and Functoriality

In the case of the classical Shimura varieties, the relation between (45) and (16) is given by "passing to components", namely we have (*cf.* [25])

$$\pi_0(Sh(\mathrm{GL}_2, \mathbb{H}^\pm)) = Sh(\mathrm{GL}_1, \{\pm 1\}). \tag{70}$$

In fact, the operation of taking connected components of (45) is realized by the map

$$\mathrm{sign} \times \det : Sh(\mathrm{GL}_2, \mathbb{H}^\pm) \to Sh(\mathrm{GL}_1, \{\pm 1\}). \tag{71}$$

We show here in Proposition 13 that this compatibility extends to a correspondence between the noncommutative Shimura varieties (26) and (58), compatible with the time evolutions and the arithmetic structures.

We have seen that, up to Morita equivalence, the noncommutative spaces of $\mathbb{Q}$-lattices modulo commensurability (but not scaling) in dimensions $n = 1, 2$ are described, respectively, by the crossed product C^*-algebras

$$C_0(\mathbb{A}_f \times \mathrm{GL}_1(\mathbb{R})) \rtimes \mathrm{GL}_1(\mathbb{Q}), \tag{72}$$

and

$$C_0(M_2(\mathbb{A}_f) \times \mathrm{GL}_2(\mathbb{R})) \rtimes \mathrm{GL}_2(\mathbb{Q}). \tag{73}$$

The functor C_0 from locally compact spaces to C^*-algebras is contravariant, while the functor C^* from discrete groups to the associated (maximal) convolution C^*-algebra is covariant. This shows that one needs to be careful in getting functoriality for crossed products.

Thus, in order to get a morphism of crossed product algebras

$$C_0(X_1) \rtimes \Gamma_1 \to C_0(X_2) \rtimes \Gamma_2,$$

it is necessary to have a proper continuous map $\pi : X_2 \to X_1$ and a proper group homomorphism $\alpha : \Gamma_1 \to \Gamma_2$, such that

$$\pi \circ \alpha(g) = g \circ \pi \quad \forall g \in \Gamma_1. \tag{74}$$

In this case, one obtains an algebra homomorphism

$$\sum f_g \, U_g \in C_0(X_1) \rtimes \Gamma_1 \mapsto \sum (f_g \circ \pi) \, U_{\alpha(g)} \in C_0(X_2) \rtimes \Gamma_2. \tag{75}$$

When the continuous map π fails to be proper, the above formula only defines a homomorphism to the multiplier algebra

$$C_0(X_1) \rtimes \Gamma_1 \to M(C_0(X_2) \rtimes \Gamma_2).$$

In the case of the BC and the GL_2 systems, we consider the map $\alpha : \mathrm{GL}_1(\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q})$ of the form

$$\alpha(r) = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}. \tag{76}$$

We also take π to be the determinant map,

$$(\rho, u) \in M_2(\mathbb{A}_f) \times \mathrm{GL}_2(\mathbb{R}) \mapsto \pi(\rho, u) = (\det(\rho), \det(u)) \in \mathbb{A}_f \times \mathrm{GL}_1(\mathbb{R}). \tag{77}$$

We then have the following result extending the classical map of Shimura varieties (70).

Proposition 13. *Let α and π be defined as in* (76) *and* (77). *Then the following holds.*

1. *The pair (π, α) induces a morphism*

$$\rho : C_0(\mathbb{A}_f \times \mathrm{GL}_1(\mathbb{R})) \rtimes \mathrm{GL}_1(\mathbb{Q}) \to M(C_0(M_2(\mathbb{A}_f) \times \mathrm{GL}_2(\mathbb{R})) \rtimes \mathrm{GL}_2(\mathbb{Q})). \tag{78}$$

2. *The restriction of ρ to homogeneous functions of weight* 0 *induces a morphism*

$$\rho_0 : C_0(\mathbb{A}_f \times \{\pm 1\}) \rtimes \mathrm{GL}_1(\mathbb{Q}) \to M(C_0(M_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}) \rtimes \mathrm{GL}_2(\mathbb{Q})). \tag{79}$$

3. *The above yields a correspondence ρ_{12} from the* GL_1*-system to the* GL_2*-system, compatible with the time evolutions and the rational subalgebras.*

Proof. 1) One checks that (74) holds, since α satisfies $\det \alpha(r) = r$. By construction, π is not proper but is clearly continuous, hence one obtains the morphism ρ to the multiplier algebra, as we discussed above.

2) The extension $\tilde{\rho}$ of ρ to multipliers gives a morphism

$$\tilde{\rho} : C(\mathbb{A}_f \times \mathrm{GL}_1(\mathbb{R})) \rtimes \mathrm{GL}_1(\mathbb{Q}) \to C(M_2(\mathbb{A}_f) \times \mathrm{GL}_2(\mathbb{R})) \rtimes \mathrm{GL}_2(\mathbb{Q}).$$

One obtains ρ_0 by restricting $\tilde{\rho}$ to homogeneous functions of weight 0 for the scaling action of $\mathbb{R}_+^*$ on $\mathrm{GL}_1(\mathbb{R})$. The image of such a function is homogeneous of weight 0 for the scaling action of $\mathbb{C}^*$ on $\mathrm{GL}_2(\mathbb{R})$ and only depends upon the sign of the modulus in $\mathbb{H}^\pm$ as in the diagram

$$\begin{array}{ccc} \mathrm{GL}_2(\mathbb{R}) & \longrightarrow & \mathbb{H}^\pm = \mathrm{GL}_2(\mathbb{R})/\mathbb{C}^* \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \mathrm{sign}} \\ \mathrm{GL}_1(\mathbb{R}) & \longrightarrow & \{\pm 1\} = \mathrm{GL}_1(\mathbb{R})/\mathbb{R}_+^*. \end{array} \tag{80}$$

3) One can restrict ρ_0 to $\mathcal{A}_1 = C(\hat{\mathbb{Z}}) \rtimes_\alpha \mathbb{N}^\times$ and one gets a morphism

$$C(\hat{\mathbb{Z}}) \rtimes_\alpha \mathbb{N}^\times \to M(C_0(M_2(\mathbb{A}_f) \times \mathbb{H}) \rtimes \mathrm{GL}_2(\mathbb{Q})^+,) \tag{81}$$

but one still needs to combine it with the Morita equivalence between $C_0(M_2(\mathbb{A}_f) \times \mathbb{H}) \rtimes \mathrm{GL}_2(\mathbb{Q})^+$ and $\mathcal{A}_2$. This Morita equivalence is given by the bimodule $\mathcal{E}$ of functions of

$$(g, \alpha, z) \in \mathcal{V} = \mathrm{GL}_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathbb{H} \tag{82}$$

invariant under the Γ action

$$(g, \alpha, z) \mapsto (g\gamma^{-1}, \gamma\, \alpha, \gamma(z)). \tag{83}$$

For $(g, \alpha, z) \in \mathcal{V}$, both $g\,\alpha \in M_2(\mathbb{A}_f)$ and $g(z) \in \mathbb{H}$ make sense, and this allows for a left action of $C_0(M_2(\mathbb{A}_f) \times \mathbb{H})$ on $\mathcal{E}$. This action is $\mathrm{GL}_2^+(\mathbb{Q})$-equivariant for the left action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $\mathcal{V}$ given by

$$h\,(g, \alpha, z) = (h\, g, \alpha, z) \quad \forall h \in \mathrm{GL}_2^+(\mathbb{Q}) \tag{84}$$

and turns $\mathcal{E}$ into a left $C_0(M_2(\mathbb{A}_f) \times \mathbb{H}) \rtimes \mathrm{GL}_2(\mathbb{Q})^+$ module. To obtain the right module structure over $\mathcal{A}_2$, one uses the convolution product

$$(\xi * f)\,(g, \alpha, z) = \sum_{s \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q})} \xi(gs^{-1}, s\,\alpha, s(z))\, f(s, \alpha, z). \tag{85}$$

The left action of $C_0(M_2(\mathbb{A}_f) \times \mathbb{H})$ extends to its multipliers and one can use (81) to obtain on $\mathcal{E}$ the required structure of left $C(\hat{\mathbb{Z}}) \rtimes_\alpha \mathbb{N}^\times$-module and right $\mathcal{A}_2$-module. One checks that this is compatible with the time evolutions σ_t. It is also compatible with the rational subalgebras $\mathcal{A}_{1,\mathbb{Q}}$ and $\mathcal{A}_{2,\mathbb{Q}}$. This compatibility can be seen using the cyclotomic condition defined in (62) and in Definition 11. ◻

The compatibility between the BC system and the CM system for imaginary quadratic fields is discussed in Proposition 22.

4 Quantum Statistical Mechanics for Imaginary Quadratic Fields

In the Kronecker–Weber case, the maximal abelian extension of $\mathbb{Q}$ is generated by the values of the exponential function at the torsion points $\mathbb{Q}/\mathbb{Z}$ of the group $\mathbb{C}/\mathbb{Z} = \mathbb{C}^*$.

Similarly, it is well known that the maximal abelian extension of an imaginary quadratic field K is generated by the values of a certain analytic function, the Weierstrass $\wp$-function, at the torsion points E_{tors} of an elliptic curve E (with complex multiplication by $\mathcal{O}$). It contains the j-invariant $j(E)$ of E. To see this, let e_k, for $k = 1, 2, 3$, denote the three values taken by the Weierstrass $\wp$-function on the set $E[2]$ of 2-torsion points of the elliptic curve E. Then $j(E)$ is obtained as a function of the e_k by the formula $j(E) = 256(1 - f + f^2)^3/(f^2(1 - f)^2)$, where $f = (e_2 - e_3)/(e_1 - e_3)$.

Thus, in the case of imaginary quadratic fields, the theory of complex multiplication of elliptic curves provides a description of K^{ab}. The ideal class group $\mathrm{Cl}(\mathcal{O})$ is naturally isomorphic to $\mathrm{Gal}(K(j)/K)$, where $K(j(E))$ is the Hilbert class field of K, *i.e.* , its maximal abelian unramified extension. In the case that $\mathrm{Cl}(\mathcal{O})$ is trivial, the situation of the CM case is exactly as for the field $\mathbb{Q}$, with $\hat{\mathcal{O}}^*$ replacing $\hat{\mathbb{Z}}^*$. Namely, in the case of class number one, the class field theory isomorphism reduces to an isomorphism $\mathrm{Gal}(K^{ab}/K) \cong \hat{\mathcal{O}}^*/\mathcal{O}^*$, which is a direct generalization of $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^*$.

The construction of BC [3] was partially generalized to other global fields (*i.e.* number fields and function fields) in [20, 16, 5]. The construction of [16] involves replacing the ring $\mathcal{O}$ of integers of K by a localized ring $\mathcal{O}_S$ which is principal and then taking a cross product of the form

$$C^*(K/\mathcal{O}_S) \rtimes \mathcal{O}_+^\times \tag{86}$$

where $\mathcal{O}_+^\times$ is the sub semi-group of K^* generated by the generators of prime ideals of $\mathcal{O}_S$. The symmetry group is $\hat{\mathcal{O}}_S^*$ and does not coincide with what is needed for class field theory except when the class number is 1. The construction of [5] involves a cross product of the form

$$C^*(K/\mathcal{O}) \rtimes J^+ \tag{87}$$

where J^+ is a suitable adelic lift of the quotient group $\mathbb{I}_K/\hat{\mathcal{O}}^*$. It gives the right partition function namely the Dedekind zeta function but not the expected symmetries. The construction of [20] involves the algebra

$$C^*(K/\mathcal{O}) \rtimes \mathcal{O}^\times \cong C(\hat{\mathcal{O}}) \rtimes \mathcal{O}^\times \tag{88}$$

and has symmetry group $\hat{\mathcal{O}}^*$, while what is needed for class field theory is a system with symmetry group $\mathbb{I}_K/K^*$. As one can see from the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{\mathcal{O}}^*/\mathcal{O}^* & \longrightarrow & \mathbb{I}_K/K^* & \longrightarrow & \mathrm{Cl}(\mathcal{O}) & \longrightarrow & 1 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\ 1 & \longrightarrow & \mathrm{Gal}(K^{ab}/K(j)) & \longrightarrow & \mathrm{Gal}(K^{ab}/K) & \longrightarrow & \mathrm{Gal}(K(j)/K) & \longrightarrow & 1, \end{array} \tag{89}$$

the action of $\hat{\mathcal{O}}^*$ is sufficient only in the case when the class number is one. In order to avoid the class number one restriction in extending the results of [3] to imaginary quadratic fields, it is natural to consider the universal situation: the moduli space of elliptic curves with level structure, *i.e.* , the *modular tower*. Using the generalization of the BC case to GL_2 constructed in [10], we will now describe the CM system constructed in [11]. This does in fact have the right properties to recover the explicit class field theory of imaginary quadratic fields from KMS states. It is based on the geometric notions of K-lattice and commensurability and extends to quadratic fields the reinterpretation of the BC system which was given in [10] in terms of $\mathbb{Q}$-lattices. We show in this section that the CM system we obtained in [11] is, in fact, Morita equivalent to the one of [20]. The two main new ingredients in our construction are the choice of a natural rational subalgebra on which to evaluate the KMS_∞ states and the fact that the group of automorphisms $\hat{\mathcal{O}}^*/\mathcal{O}^*$ should be enriched by further symmetries, this time given by *endomorphisms*, so that the actual group of symmetries of the system is exactly $\mathbb{I}_K/K^*$. In particular, the choice of the rational subalgebra differs from [20], hence, even though Morita equivalent, the systems are inequivalent as "noncommutative pro-varieties over $\mathbb{Q}$".

4.1 *K*-Lattices and Commensurability

In order to compare the BC system, the GL_2 system and the CM case, we give a definition of K-lattices, for K an imaginary quadratic field. The quantum statistical mechanical system we shall construct to recover the explicit class field theory of imaginary quadratic fields will be related to commensurability of 1-dimensional K-lattices. This will be analogous to the description of the BC system in terms of commensurability of 1-dimensional $\mathbb{Q}$-lattices. On the other hand, since there is a forgetful map from 1-dimensional K-lattices to 2-dimensional $\mathbb{Q}$-lattices, we will also be able to treat the CM case as a specialization of the GL_2 system at CM points.

Let $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\tau$ be the ring of integers of an imaginary quadratic field $K = \mathbb{Q}(\tau)$; fix the embedding $K \hookrightarrow \mathbb{C}$ so that $\tau \in \mathbb{H}$. Note that $\mathbb{C}$ then becomes a K-vector space and in particular an $\mathcal{O}$-module. The choice of τ as above also determines an embedding

$$q_\tau : K \hookrightarrow M_2(\mathbb{Q}). \tag{90}$$

The image of its restriction $q_\tau : K^* \hookrightarrow \mathrm{GL}_2^+(\mathbb{Q})$ is characterized by the property that (*cf.* [32] Proposition 4.6)

$$q_\tau(K^*) = \{g \in \mathrm{GL}_2^+(\mathbb{Q}) : \ g(\tau) = \tau\}. \tag{91}$$

For $g = q_\tau(x)$ with $x \in K^*$, we have $\det(g) = \mathfrak{n}(x)$, where $\mathfrak{n} : K^* \to \mathbb{Q}^*$ is the norm map.

Definition 14. *For K an imaginary quadratic field, a 1-dimensional K-lattice (Λ, ϕ) is a finitely generated $\mathcal{O}$-submodule $\Lambda \subset \mathbb{C}$, such that $\Lambda \otimes_{\mathcal{O}} K \cong K$, together with a morphism of $\mathcal{O}$-modules*

$$\phi : K/\mathcal{O} \to K\Lambda/\Lambda. \tag{92}$$

A 1-dimensional K-lattice is invertible *if ϕ is an isomorphism of $\mathcal{O}$-modules.*

Notice that in the definition we assume that the $\mathcal{O}$-module structure is compatible with the embeddings of both $\mathcal{O}$ and Λ in $\mathbb{C}$.

Lemma 15. *A 1-dimensional K-lattice is, in particular, a 2-dimensional $\mathbb{Q}$-lattice. Moreover, as an $\mathcal{O}$-module, Λ is projective.*

Proof. First notice that $K\Lambda = \mathbb{Q}\Lambda$, since $\mathbb{Q}\mathcal{O} = K$. This, together with $\Lambda \otimes_{\mathcal{O}} K \cong K$, shows that the $\mathbb{Q}$-vector space $\mathbb{Q}\Lambda$ is 2-dimensional. Since $\mathbb{R}\Lambda = \mathbb{C}$, and Λ is finitely generated as an abelian group, this shows that Λ is a lattice. The basis $\{1, \tau\}$ provides an identification of $K/\mathcal{O}$ with $\mathbb{Q}^2/\mathbb{Z}^2$, so that we can view ϕ as a homomorphism of abelian groups $\phi : \mathbb{Q}^2/\mathbb{Z}^2 \to \mathbb{Q}\Lambda/\Lambda$. The pair (Λ, ϕ) thus gives a two dimensional $\mathbb{Q}$-lattice.

As an $\mathcal{O}$-module Λ is isomorphic to a finitely generated $\mathcal{O}$-submodule of K, hence to an ideal in $\mathcal{O}$. Every ideal in a Dedekind domain $\mathcal{O}$ is finitely generated projective over $\mathcal{O}$. ☐

Suppose given a 1-dimensional K-lattice (Λ, ϕ). Then the elliptic curve $E = \mathbb{C}/\Lambda$ has complex multiplication by K, namely there is an isomorphism

$$\iota : K \xrightarrow{\cong} \mathrm{End}(E) \otimes \mathbb{Q}. \tag{93}$$

In general, for Λ a lattice in $\mathbb{C}$, if the elliptic curve $E = \mathbb{C}/\Lambda$ has complex multiplication (*i.e.* there is an isomorphism (93) for K an imaginary quadratic field), then the endomorphisms of E are given by $\mathrm{End}(E) = \mathcal{O}_\Lambda$, where $\mathcal{O}_\Lambda$ is the order of Λ,

$$\mathcal{O}_\Lambda = \{x \in K : \ x\Lambda \subset \Lambda\}. \tag{94}$$

Notice that the elliptic curves $E = \mathbb{C}/\Lambda$, where Λ is a 1-dimensional K-lattice, have $\mathcal{O}_\Lambda = \mathcal{O}$, the maximal order. The number of distinct isomorphism classes of elliptic curves E with $\mathrm{End}(E) = \mathcal{O}$ is equal to the class number h_K. All the other elliptic curves with complex multiplication by K are obtained from these by isogenies.

Definition 16. *Two 1-dimensional K-lattices (Λ_1, ϕ_1) and (Λ_2, ϕ_2) are commensurable if $K\Lambda_1 = K\Lambda_2$ and $\phi_1 = \phi_2$ modulo $\Lambda_1 + \Lambda_2$.*

One checks as in the case of $\mathbb{Q}$-lattices (*cf.* [10]) that it is an equivalence relation.

Lemma 17. *Two 1-dimensional K-lattices are commensurable iff the underlying $\mathbb{Q}$-lattices are commensurable. Up to scaling, any K-lattice Λ is equivalent to a K-lattice $\Lambda' = \lambda\Lambda \subset K \subset \mathbb{C}$, for a $\lambda \in \mathbb{C}^*$. The lattice Λ' is unique modulo K^*.*

Proof. The first statement holds, since for 1-dimensional K-lattices we have $K\Lambda = \mathbb{Q}\Lambda$. For the second statement, the K-vector space $K\Lambda$ is 1-dimensional. If ξ is a generator, then $\xi^{-1}\Lambda \subset K$. The remaining ambiguity is only by scaling by elements in K^*. □

A more explicit description of the space of 1-dimensional K-lattices, the commensurability relation, and the action of $\mathbb{C}^*$ by scaling is the following (*cf.* [11]).

Proposition 18. *The following properties hold.*

1. *The data (Λ, ϕ) of a 1-dimensional K-lattice are equivalent to data (ρ, s) of an element $\rho \in \hat{\mathcal{O}}$ and $s \in \mathbb{A}_K^*/K^*$, modulo the $\hat{\mathcal{O}}^*$-action given by $(\rho, s) \mapsto (x^{-1}\rho, xs)$, $x \in \hat{\mathcal{O}}^*$. Thus, the space of 1-dimensional K-lattices is given by*
$$\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}_K^*/K^*). \tag{95}$$
2. *Let $\mathbb{A}_K^{\cdot} = \mathbb{A}_{K,f} \times \mathbb{C}^*$ be the subset of adèles of K with nontrivial archimedean component. The map $\Theta(\rho, s) = \rho\, s$,*
$$\Theta : \hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}_K^*/K^*) \to \mathbb{A}_K^{\cdot}/K^*, \tag{96}$$
preserves commensurability and induces an identification of the set of commensurability classes of 1-dimensional K-lattices (not up to scale) with the space $\mathbb{A}_K^{\cdot}/K^*$.
3. *The map defined as $\Upsilon : (\Lambda, \phi) \mapsto \rho \in \hat{\mathcal{O}}/K^*$ for principal K-lattices extends to an identification, given by $\Upsilon : (\Lambda, \phi) \mapsto s_f\, \rho \in \mathbb{A}_{K,f}/K^*$, of the set of commensurability classes of 1-dimensional K-lattices up to scaling with the quotient*
$$\hat{\mathcal{O}}/K^* = \mathbb{A}_{K,f}/K^*. \tag{97}$$

The proof (*cf.* [11]) uses the fact that the $\mathcal{O}$-module Λ can be described in the form $\Lambda_s = s_\infty^{-1}(s_f\hat{\mathcal{O}} \cap K)$, where $s = (s_f, s_\infty) \in \mathbb{A}_K^*$. This satisfies $\Lambda_{ks} = \Lambda_s$ for all $k \in (\hat{\mathcal{O}}^* \times 1)\, K^* \subset \mathbb{A}_K^*$, so that, up to scaling, Λ can be identified with an ideal in $\mathcal{O}$, written in the form $s_f\hat{\mathcal{O}} \cap K$ (*cf.* [32] §5.2).

The quotient $\mathbb{A}_{K,f}/K^*$ has a description in terms of elliptic curves, analogous to what we explained in the case of the GL_2-system. In fact, we can associate to (Λ, ϕ) the data (E, η) of an elliptic curve $E = \mathbb{C}/\Lambda$ with complex multiplication (93), such that the embedding $K \hookrightarrow \mathbb{C}$ determined by this identification and by the action of $\mathrm{End}(E)$ on the tangent space of E at the origin is the embedding specified by τ (*cf.* [25] p.28, [32] §5.1), and an $\mathbb{A}_{K,f}$-homomorphism

$$\eta : \mathbb{A}_{K,f} \to \Lambda \otimes_{\mathcal{O}} \mathbb{A}_{K,f}, \tag{98}$$

The composite map

$$\mathbb{A}_{K,f} \xrightarrow{\eta} \Lambda \otimes_{\mathcal{O}} \mathbb{A}_{K,f} \xrightarrow{\simeq} \mathbb{A}_{K,f},$$

determines an element $s_f\, \rho \in \mathbb{A}_{K,f}$. The set of equivalence classes of data (E, ι, η), where equivalence is given by an isogeny of the elliptic curve compatible with the other data, is the quotient $\mathbb{A}_{K,f}/K^*$.

This generalizes to the non-invertible case the analogous result for invertible 1-dimensional K-lattices (data (E, ι, η), with η an isomorphism realized by an element $\rho \in \mathbb{A}^*_{K,f}$) treated in [25], where the set of equivalence classes is given by the idèle class group of the imaginary quadratic field,

$$\mathbb{I}_K/K^* = \mathrm{GL}_1(K)\backslash \mathrm{GL}_1(\mathbb{A}_{K,f}) = C_K/D_K. \tag{99}$$

4.2 Algebras of Coordinates

We now describe the noncommutative algebra of coordinates of the space of commensurability classes of 1-dimensional K-lattices up to scaling.

To this purpose, we first consider the groupoid $\tilde{\mathcal{R}}_K$ of the equivalence relation of commensurability on 1-dimensional K-lattices (not up to scaling). By construction, this groupoid is a subgroupoid of the groupoid $\tilde{\mathcal{R}}$ of commensurability classes of 2-dimensional $\mathbb{Q}$-lattices. Its structure as a locally compact étale groupoid is inherited from this embedding.

The groupoid $\tilde{\mathcal{R}}_K$ corresponds to the quotient $\mathbb{A}^\cdot_K/K^*$. Its C^*-algebra is given up to Morita equivalence by the crossed product

$$C_0(\mathbb{A}^\cdot_K) \rtimes K^*. \tag{100}$$

The case of commensurability classes of 1-dimensional K-lattices up to scaling is more delicate. In fact, Proposition 18 describes set theoretically the space of commensurability classes of 1-dimensional K-lattices up to scaling as the quotient $\mathbb{A}_{K,f}/K^*$. This has a noncommutative algebra of coordinates, which is the crossed product

$$C_0(\mathbb{A}_{K,f}/\mathcal{O}^*) \rtimes K^*/\mathcal{O}^*. \tag{101}$$

As we are going to show below, this is Morita equivalent to the noncommutative algebra $\mathcal{A}_K = C^*(G_K)$ obtained by taking the quotient by scaling $G_K = \tilde{\mathcal{R}}_K/\mathbb{C}^*$ of the groupoid of the equivalence relation of commensurability.

Unlike what happens when taking the quotient by the scaling action of $\mathbb{C}^*$ in the GL_2-system, in the CM case the quotient $G_K = \tilde{\mathcal{R}}_K/\mathbb{C}^*$ is still a groupoid.

The simplest way to check this is to write $\tilde{\mathcal{R}}_K$ as the union of the two groupoids $\tilde{\mathcal{R}}_K = G_0 \cup G_1$, corresponding respectively to pairs of commensurable K-lattices (L, L') with $L = (\Lambda, 0), L' = (\Lambda', 0)$ and (L, L') with $L = (\Lambda, \phi \neq 0), L' = (\Lambda', \phi' \neq 0)$. The scaling action of $\mathbb{C}^*$ on G_0 is the identity on $\mathcal{O}^*$ and the corresponding action of $\mathbb{C}^*/\mathcal{O}^*$ is free on the units of G_0. Thus, the quotient $G_0/\mathbb{C}^*$ is a groupoid. Similarly, the action of $\mathbb{C}^*$ on G_1 is free on the units of G_1 and the quotient $G_1/\mathbb{C}^*$ is a groupoid.

The quotient topology turns G_K into a locally compact étale groupoid. The algebra of coordinates $\mathcal{A}_K = C^*(G_K)$ is described equivalently by restricting the convolution product of the algebra of $\tilde{\mathcal{R}}_K$ to weight zero functions with $\mathbb{C}^*$-compact support and then passing to the C^* completion, as in the GL_2-case. In other words the algebra $\mathcal{A}_K$ of the CM system is the convolution algebra of weight zero functions on the groupoid $\tilde{\mathcal{R}}_K$ of the equivalence relation of commensurability on K-lattices. Elements in the algebra are functions of pairs $(\Lambda, \phi), (\Lambda', \phi')$ of commensurable 1-dimensional K-lattices satisfying, for all $\lambda \in \mathbb{C}^*$,

$$f(\lambda(\Lambda, \phi), \lambda(\Lambda', \phi')) = f((\Lambda, \phi), (\Lambda', \phi')).$$

Let us compare the setting of (101) *i.e.* the groupoid $\mathbb{A}_{K,f}/\mathcal{O}^* \rtimes K^*/\mathcal{O}^*$ with the groupoid G_K. In fact, the difference between these two settings can be seen by looking at the case of K-lattices with $\phi = 0$. In the first case, this corresponds to the point $0 \in \mathbb{A}_{K,f}/\mathcal{O}^*$, which has stabilizer $K^*/\mathcal{O}^*$, hence we obtain the group C^*-algebra of $K^*/\mathcal{O}^*$. In the other case, the corresponding groupoid is obtained as a quotient by $\mathbb{C}^*$ of the groupoid $\tilde{\mathcal{R}}_{K,0}$ of pairs of commensurable $\mathcal{O}$-modules (finitely generated of rank one) in $\mathbb{C}$. In this case the units of the groupoid $\tilde{\mathcal{R}}_{K,0}/\mathbb{C}^*$ can be identified with the elements of $\mathrm{Cl}(\mathcal{O})$ and the reduced groupoid by any of these units is the group $K^*/\mathcal{O}^*$. The general result below gives the Morita equivalence in general.

Proposition 19. *Let* $\mathcal{A}_{K,princ} := C_0(\mathbb{A}_{K,f}/\mathcal{O}^*) \rtimes K^*/\mathcal{O}^*$. *Let* $\mathcal{H}$ *be the space of pairs up to scaling* $((\Lambda, \phi), (\Lambda', \phi'))$ *of commensurable* K*-lattices, with* (Λ, ϕ) *principal. The space* $\mathcal{H}'$ *is defined analogously with* (Λ', ϕ') *principal. Then* $\mathcal{H}$ *(resp.* $\mathcal{H}'$*) has the structure of* $\mathcal{A}_{K,princ}$*–*$\mathcal{A}_K$ *(resp.* $\mathcal{A}_K$*–*$\mathcal{A}_{K,princ}$*) bimodule. These bimodules give a Morita equivalence between the algebras* $\mathcal{A}_{K,princ}$ *and* $\mathcal{A}_K = C^*(G_K)$.

Proof. The correspondence given by these bimodules has the effect of reducing to the principal case. In that case the groupoid of the equivalence relation (not up to scaling) is given by the crossed product $\mathbb{A}^\cdot_K \rtimes K^*$. When taking the quotient by $\mathbb{C}^*$ we then obtain the groupoid $\mathbb{A}_{K,f}/\mathcal{O}^* \rtimes K^*/\mathcal{O}^*$. □

Recall that the C^*-algebra for the GL_2-system is not unital, the reason being that the space of 2-dimensional $\mathbb{Q}$-lattices up to scaling is noncompact, due to the presence of the modulus $z \in \mathbb{H}$ of the lattice. When restricting to 1-dimensional K-lattices up to scaling, this parameter z affects only finitely many values, corresponding to representatives $\Lambda = \mathbb{Z} + \mathbb{Z}z$ of the classes in $\mathrm{Cl}(\mathcal{O})$.

Thus, the algebra $\mathcal{A}_K$ is unital.

In terms of the groupoid, this can be seen from the fact that the set $G_K^{(0)}$ of units of G_K is the compact space

$$X = G_K^{(0)} = \hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}_{K,f}^*/K^*). \tag{102}$$

obtained as the quotient of $\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}_K^*/K^*)$ by the action of $\mathbb{C}^*$. Notice that $\mathbb{A}_{K,f}^*/(K^* \times \hat{\mathcal{O}}^*)$ is $\mathrm{Cl}(\mathcal{O})$.

By restriction from the GL_2-system, there is a homomorphism $\mathfrak{n}$ from the groupoid $\tilde{\mathcal{R}}_K$ to $\mathbb{R}_+^*$ given by the covolume of a commensurable pair of K-lattices. More precisely given such a pair $(L, L') = ((\Lambda, \phi), (\Lambda', \phi'))$ we let

$$|L/L'| = \mathrm{covolume}(\Lambda')/\mathrm{covolume}(\Lambda) \tag{103}$$

This is invariant under scaling both lattices, so it is defined on $G_K = \tilde{\mathcal{R}}_K/\mathbb{C}^*$. Up to scale, we can identify the lattices in a commensurable pair with ideals in $\mathcal{O}$. The covolume is then given by the ratio of the norms. This defines a time evolution on the algebra $\mathcal{A}_K$ by

$$\sigma_t(f)(L, L') = |L/L'|^{it} f(L, L'). \tag{104}$$

We construct representations for the algebra $\mathcal{A}_K$. For an étale groupoid G_K, every unit $y \in G_K^{(0)}$ defines a representation π_y by left convolution of the algebra of G_K in the Hilbert space $\mathcal{H}_y = \ell^2((G_K)_y)$, where $(G_K)_y$ is the set of elements with source y. The representations corresponding to points that have a nontrivial automorphism group will no longer be irreducible. As in the GL_2-case, this defines the norm on $\mathcal{A}_K$ as

$$\|f\| = \sup_y \|\pi_y(f)\|. \tag{105}$$

Notice that, unlike in the case of the GL_2-system, we are dealing here with amenable groupoids, hence the distinction between the maximal and the reduced C^*-algebra does not arise.

The following result of [11] shows the relation to the Dedekind zeta function.

Lemma 20. *1. Given an invertible K-lattice (Λ, ϕ), the map*

$$(\Lambda', \phi') \mapsto J = \{x \in \mathcal{O} | x\Lambda' \subset \Lambda\} \tag{106}$$

gives a bijection of the set of K-lattices commensurable to (Λ, ϕ) with the set of ideals in $\mathcal{O}$.

2. Invertible K-lattices define positive energy representations.
3. The partition function is the Dedekind zeta function $\zeta_K(\beta)$ of K.

As in Theorem 1.26 and Lemma 1.27 of [10] in the GL_2-case, one uses the fact that, if Λ is an invertible 2-dimensional $\mathbb{Q}$-lattice and Λ' is commensurable to Λ, then $\Lambda \subset \Lambda'$. The map above is well defined, since $J \subset \mathcal{O}$ is an ideal. Moreover, $J\Lambda' = \Lambda$, since $\mathcal{O}$ is a Dedekind domain. It is shown in [11] that the map is both injective and surjective. We use the notation

$$J^{-1}(\Lambda, \phi) = (\Lambda', \phi). \tag{107}$$

For an invertible K-lattice, this gives an identification of $(G_K)_y$ with the set $\mathcal{J}$ of ideals $J \subset \mathcal{O}$. The covolume is then given by the norm. The corresponding Hamiltonian is of the form

$$H\,\epsilon_J = \log \mathfrak{n}(J)\ \epsilon_J, \tag{108}$$

with non-negative eigenvalues, hence the partition function is the Dedekind zeta function

$$Z(\beta) = \sum_{J \text{ ideal in } \mathcal{O}} \mathfrak{n}(J)^{-\beta} = \zeta_K(\beta). \tag{109}$$

4.3 Symmetries

We shall now adapt the discussion of symmetries of the GL(2) system to K-lattices, adopting a contravariant notation instead of the covariant one used in [10].

Proposition 21. *The semigroup $\hat{\mathcal{O}} \cap \mathbb{A}^*_{K,f}$ acts on the algebra $\mathcal{A}_K$ by endomorphisms. The subgroup $\hat{\mathcal{O}}^*$ acts on $\mathcal{A}_K$ by automorphisms. The subsemigroup $\mathcal{O}^\times$ acts by inner endomorphisms.*

Consider the set of K-lattices (Λ, ϕ) such that ϕ is well defined modulo $J\Lambda$, for an ideal $J = s\hat{\mathcal{O}} \cap K$. Namely, the map $\phi : K/\mathcal{O} \longrightarrow K\Lambda/\Lambda$ factorises as $K/\mathcal{O} \longrightarrow K\Lambda/J\Lambda \to K\Lambda/\Lambda$. We say, in this case, that the K-lattice (Λ, ϕ) is divisible by J. For a commensurable pair (Λ, ϕ) and (Λ', ϕ'), and an element $f \in \mathcal{A}_K$, we set

$$\theta_s(f)((\Lambda, \phi), (\Lambda', \phi')) = \begin{cases} f((\Lambda, s^{-1}\phi), (\Lambda', s^{-1}\phi')) & \text{both } K\text{-lattices are divisible by } J \\ 0 & \text{otherwise.} \end{cases} \tag{110}$$

Formula (110) defines an endomorphism of $\mathcal{A}_K$ with range the algebra reduced by an idempotent e_J. Clearly, for $s \in \hat{\mathcal{O}}^*$, the above defines an automorphism, which is compatible with the time evolution. For $s \in \mathcal{O}^\times$, the endomorphism (110) is inner,

$$\theta_s(f) = \mu_s\, f\, \mu_s^*,$$

with $\mu_s \in \mathcal{A}_K$ given by

$$\mu_s((\Lambda,\phi),(\Lambda',\phi')) = \begin{cases} 1 & \Lambda = s^{-1}\Lambda' \text{ and } \phi' = \phi \\ 0 & \text{otherwise.} \end{cases} \tag{111}$$

The range of μ_s is the projection e_J, with J the principal ideal generated by s. The isometries μ_s are eigenvectors of the time evolution, namely $\sigma_t(\mu_s) = \mathfrak{n}(s)^{it}\,\mu_s$.

Recall that $\mathbb{A}_{K,f} = \hat{\mathcal{O}}.K^*$. Thus, we can pass to the corresponding group of symmetries, modulo inner, which is given by the idèle class group $\mathbb{A}^*_{K,f}/K^*$, which is identified, by the class field theory isomorphism, with the Galois group $\mathrm{Gal}(K^{ab}/K)$.

This shows that we have an action of the idèle class group on the set of extremal KMS_β states of the CM system. The action of the subgroup $\hat{\mathcal{O}}^*/\mathcal{O}^*$ is by automorphisms, while the action of the quotient group $\mathrm{Cl}(\mathcal{O})$ is by endomorphisms, as we expected according to diagram (89).

4.4 Comparison with Other Systems

First we investigate the functoriality issue for the CM-system associated to a quadratic imaginary extension K of $\mathbb{Q}$ as above. This describes the compatibility between the CM system and the BC system.

By Proposition 19 we can replace the C^*-algebra $\mathcal{A}_K$ by the Morita equivalent one $\mathcal{A}_{K,princ} = C_0(\mathbb{A}_{K,f}/\mathcal{O}^*) \rtimes K^*/\mathcal{O}^*$. We need to relate it to the BC-system *i.e.* again, up to Morita equivalence, to the crossed product algebra $C_0(\mathbb{A}_f) \rtimes \mathrm{GL}_1(\mathbb{Q})^+$.

Let $N : \mathbb{A}_{K,f} \to \mathbb{A}_f$ be the norm map and $\iota : N(K^*) \to K^*$ be a group homomorphism section of $N : K^* \to N(K^*)$.

Proposition 22. *The pair (N,ι) induces a morphism*

$$\varrho \;:\; C_0(\mathbb{A}_f) \rtimes N(K^*) \to M(C_0(\mathbb{A}_{K,f}/\mathcal{O}^*) \rtimes K^*/\mathcal{O}^*) = M(\mathcal{A}_{K,princ})$$

to the multiplier algebra of $\mathcal{A}_{K,princ}$.

Proof. This is an immediate application of the discussion in Section 3.7. In fact, one checks that condition (74) is fulfilled by construction. Note that, in the setup as above, $N(K^*)$ is a subgroup of $\mathrm{GL}_1(\mathbb{Q})^+$ and, moreover, $N(\mathcal{O}^*) = 1$. The noncommutative space $\mathbb{A}_f/N(K^*)$, quotient of $\mathbb{A}_f$ by $N(K^*)$, is an etale infinite covering of the space of one dimensional $\mathbb{Q}$-lattices up to scaling. ◻

It is also useful to see explicitly the relation of the algebra $\mathcal{A}_K$ of the CM system to the algebras previously considered in generalizations of the

Bost–Connes results, especially those of [5], [16], and [20]. This will explain why the algebra $\mathcal{A}_K$ contains exactly the amount of extra information to allow for the full explict class field theory to appear.

The partition function of the system considered in [16] agrees with the Dedekind zeta function only in the case of class number one. A different system, which has partition function the Dedekind zeta function in all cases, was introduced in [5]. Our system also has as partition function the Dedekind zeta function, independently of class number. It however differs from the system of [5]. In fact, in the latter, which is a semigroup crossed product, the natural quotient of the C^*-algebra obtained by specializing at the fixed point of the semigroup is the group ring of an extension of the class group $\mathrm{Cl}(\mathcal{O})$ by $K^*/\mathcal{O}^*$, while in our case, when specializing similarly to the K-lattices with $\phi = 0$, we obtain an algebra Morita equivalent to the group ring of $K^*/\mathcal{O}^*$. Thus, the two systems are not naturally Morita equivalent.

The system considered in [20] is analyzed there only under the hypothesis of class number one. It can be recovered from our system, which has no restrictions on class number, by reduction to those K-lattices that are principal. Thus, the system of [20] is Morita equivalent to our system (*cf.* Proposition 19).

Notice, moreover, that the crossed product algebra $C(\hat{\mathcal{O}}) \rtimes \mathcal{O}^\times$ considered in some generalizations of the BC system is more similar to the "determinant part" of the GL_2-system (*cf.* Section 1.7 of [10]), namely to the algebra $C(M_2(\hat{\mathbb{Z}})) \rtimes M_2^+(\mathbb{Z})$, than to the full GL_2-system.

5 KMS States and Complex Multiplication

We can now describe the arithmetic subalgebra $\mathcal{A}_{K,\mathbb{Q}}$ of $\mathcal{A}_K$. The relation between the CM and the GL_2-system provides us with a natural choice for $\mathcal{A}_{K,\mathbb{Q}}$.

Definition 23. *The algebra $\mathcal{A}_{K,\mathbb{Q}}$ is the K-algebra obtained by*

$$\mathcal{A}_{K,\mathbb{Q}} = \mathcal{A}_{2,\mathbb{Q}}|_{G_K} \otimes_{\mathbb{Q}} K. \tag{112}$$

Here $\mathcal{A}_{2,\mathbb{Q}}|_{G_K}$ denotes the $\mathbb{Q}$-algebra obtained by restricting elements of the algebra $\mathcal{A}_{2,\mathbb{Q}}$ of Definition 11 to the $\mathbb{C}^*$-quotient G_K of the subgroupoid $\tilde{\mathcal{R}}_K \subset \tilde{\mathcal{R}}_2$.

Notice that, for the CM system, $\mathcal{A}_{K,\mathbb{Q}}$ is a subalgebra of $\mathcal{A}_K$, not just a subalgebra of unbounded multipliers as in the GL_2-system, because of the fact that $\mathcal{A}_K$ is unital.

We are now ready to state the main result of the CM case. The following theorem gives the structure of KMS states for the system $(\mathcal{A}_K, \sigma_t)$ and shows that this system gives a solution to Problem 1 for the imaginary quadratic field K.

Theorem 24. (Connes–Marcolli–Ramachandran [11]) *Consider the system $(\mathcal{A}_K, \sigma_t)$ described in the previous section. The extremal KMS states of this system satisfy:*

- *In the range $0 < \beta \leq 1$ there is a unique KMS state.*
- *For $\beta > 1$, extremal KMS$_\beta$ states are parameterized by invertible K-lattices,*
$$\mathcal{E}_\beta \simeq \mathbb{A}^*_{K,f}/K^* \tag{113}$$
with a free and transitive action of $C_K/D_K \cong \mathbb{A}^*_{K,f}/K^*$ as symmetries.
- *In this range, the extremal KMS$_\beta$ state associated to an invertible K-lattice $L = (\Lambda, \phi)$ is of the form*
$$\varphi_{\beta,L}(f) = \zeta_K(\beta)^{-1} \sum_{J \text{ ideal in } \mathcal{O}} f(J^{-1}L, J^{-1}L)\, \mathfrak{n}(J)^{-\beta}, \tag{114}$$
where $\zeta_K(\beta)$ is the Dedekind zeta function, and $J^{-1}L$ defined as in (107).
- *The set of extremal KMS$_\infty$ states (as weak limits of KMS$_\beta$ states) is still given by* (113).
- *The extremal KMS$_\infty$ states $\varphi_{\infty,L}$ of the CM system, evaluated on the arithmetic subalgebra $\mathcal{A}_{K,\mathbb{Q}}$, take values in K^{ab}, with $\varphi_{\infty,L}(\mathcal{A}_{K,\mathbb{Q}}) = K^{ab}$.*
- *The class field theory isomorphism* (2) *intertwines the action of $\mathbb{A}^*_{K,f}/K^*$ by symmetries of the system $(\mathcal{A}_K, \sigma_t)$ and the action of* $\mathrm{Gal}(K^{ab}/K)$ *on the image of $\mathcal{A}_{K,\mathbb{Q}}$ under the extremal KMS$_\infty$ states. Namely, for all $\varphi_{\infty,L} \in \mathcal{E}_\infty$ and for all $f \in \mathcal{A}_{K,\mathbb{Q}}$,*
$$\alpha(\varphi_{\infty,L}(f)) = (\varphi_{\infty,L} \circ \theta^{-1}(\alpha))(f), \qquad \forall \alpha \in \mathrm{Gal}(K^{ab}/K). \tag{115}$$

Notice that the result stated above is substantially different from the GL_2-system. This is not surprising, as the following general fact illustrates. Given an étale groupoid $\mathcal{G}$ and a full subgroupoid $\mathcal{G}' \subset \mathcal{G}$, let ρ be a homomorphism $\rho : \mathcal{G} \to \mathbb{R}^*_+$. The inclusion $\mathcal{G}' \subset \mathcal{G}$ gives a correspondence between the C^*-algebras associated to $\mathcal{G}'$ and $\mathcal{G}$, compatible with the time evolution associated to ρ and its restriction to $\mathcal{G}'$. The following simple example, however, shows that, in general, the KMS states for the $\mathcal{G}'$ system do not map to KMS states for the $\mathcal{G}$ system. We let $\mathcal{G}$ be the groupoid with units $\mathcal{G}^{(0)}$ given by an infinite countable set, and morphisms given by all pairs of units. Consider a finite subset of $\mathcal{G}^{(0)}$ and let $\mathcal{G}'$ be the reduced groupoid. Finally, let ρ be trivial. Clearly, the $\mathcal{G}'$ system admits a KMS state for all temperatures given by the trace, while, since there is no tracial state on the compact operators, the $\mathcal{G}$ system has no KMS states.

5.1 Low Temperature KMS States and Galois Action

The partition function $Z_K(\beta)$ of (109) converges for $\beta > 1$. We have also seen in the previous section that invertible K-lattices $L = (\Lambda, \phi)$ determine positive energy representations of $\mathcal{A}_K$ on the Hilbert space $\mathcal{H} = \ell^2(\mathcal{J})$ where $\mathcal{J}$ is the set of ideals of $\mathcal{O}$. Thus, the formula

$$\varphi_{\beta,L}(f) = \frac{\mathrm{Tr}\,(\pi_L(f)\,\exp(-\beta H))}{\mathrm{Tr}(\exp(-\beta H))} \tag{116}$$

defines an extremal KMS_β state, with the Hamiltonian H of (108). These states are of the form (114). It is not hard to see that distinct elements in $\mathbb{A}_{K,f}/K^*$ define distinct states $\varphi_{\beta,L}$. This shows that we have an injection of $\mathbb{A}_{K,f}/K^* \subset \mathcal{E}_\beta$. Conversely, every extremal KMS_β state is of the form (114). This is shown in [11] by first proving that KMS_β states are given by measures on the space X of K-lattices (up to scaling),

$$\varphi(f) = \int_X f(L,L)\,d\mu(L)\,, \quad \forall f \in \mathcal{A}_K\,. \tag{117}$$

One then shows that, when $\beta > 1$ this measure is carried by the commensurability classes of invertible K-lattices. (We refer the reader to [11] for details.)

The weak limits as $\beta \to \infty$ of states in $\mathcal{E}_\beta$ define states in $\mathcal{E}_\infty$ of the form

$$\varphi_{\infty,L}(f) = f(L,L). \tag{118}$$

Some care is needed in defining the action of the symmetry group $\mathbb{A}_{K,f}/K^*$ on extremal states at zero temperature. In fact, as it happens also in the GL_2-case, for an invertible K-lattice evaluating $\varphi_{\infty,L}$ on $\theta_s(f)$ does not give a nontrivial action in the case of endomorphisms. However, there is a nontrivial action induced on $\mathcal{E}_\infty$ by the action on $\mathcal{E}_\beta$ for finite β and it is obtained as

$$\Theta_s(\varphi_{\infty,L})(f) = \lim_{\beta\to\infty} \left(W_\beta(\varphi_{\infty,L}) \circ \theta_s\right)(f), \tag{119}$$

where W_β is the "warm up" map (9). This gives

$$\Theta_s(\varphi_{\infty,L}) = \varphi_{\infty,L_s}, \tag{120}$$

with L_s the invertible K-lattice $(J_s^{-1}\Lambda, s^{-1}\phi)$.

Thus the action of the symmetry group $\mathbb{I}_K/K^*$ is given by

$$L = (\Lambda,\phi) \mapsto L_s = (J_s^{-1}\Lambda, s^{-1}\phi)\,, \quad \forall s \in \mathbb{I}_K/K^*\,. \tag{121}$$

When we evaluate states $\varphi_{\infty,L}$ on elements $f \in \mathcal{A}_{K,\mathbb{Q}}$ of the arithmetic subalgebra we obtain

$$\varphi_{\infty,L}(f) = f(L,L) = g(L)\,, \tag{122}$$

where the function g is the lattice function of weight 0 obtained as the restriction of f to the diagonal. By construction of $\mathcal{A}_{K,\mathbb{Q}}$, one obtains in this way all the evaluations $f \mapsto f(z)$ of elements of the modular field F on the finitely many modules $z \in \mathbb{H}$ of the classes of K-lattices.

The modular functions $f \in F$ that are defined at τ define a subring B of F. The theory of complex multiplication (*cf.* [32], §5) shows that the subfield $F_\tau \subset \mathbb{C}$ generated by the values $f(\tau)$, for $f \in B$, is the maximal abelian extension of K (we have fixed an embedding $K \subset \mathbb{C}$),

$$F_\tau = K^{ab}. \tag{123}$$

Moreover, the action of $\alpha \in \mathrm{Gal}(K^{ab}/K)$ on the values $f(z)$ is given by

$$\alpha f(z) = f^{\sigma q_\tau \theta^{-1}(\alpha)}(z). \tag{124}$$

In this formula the notation $f \mapsto f^\gamma$ denotes the action of an element $\gamma \in \mathrm{Aut}(F)$ on the elements $f \in F$, the map θ is the class field theory isomorphism (2), q_τ is the embedding of $\mathbb{A}^*_{K,f}$ in $\mathrm{GL}_2(\mathbb{A})$ and σ is as in the diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K^* & \xrightarrow{\iota} & \mathrm{GL}_1(\mathbb{A}_{K,f}) & \xrightarrow{\theta} & \mathrm{Gal}(K^{ab}/K) & \longrightarrow & 1 \\ & & & & \downarrow{\scriptstyle q_\tau} & & & & \\ 1 & \longrightarrow & \mathbb{Q}^* & \longrightarrow & \mathrm{GL}_2(\mathbb{A}_f) & \xrightarrow{\sigma} & \mathrm{Aut}(F) & \longrightarrow & 1. \end{array} \tag{125}$$

Thus, when we act by an element $\alpha \in \mathrm{Gal}(K^{ab}/K)$ on the values on $\mathcal{A}_{K,\mathbb{Q}}$ of an extremal KMS_∞ state we have

$$\alpha\, \varphi_{\infty,L}(f) = \varphi_{\infty,L_s}(f) \tag{126}$$

where $s = \theta^{-1}(\alpha) \in \mathbb{I}_K/K^*$.

This corresponds to the result of Theorem 1.39 of [10] for the case of 2-dimensional $\mathbb{Q}$-lattices (see equations (1.130) and following in [10]) with the slight nuance that we used there a covariant notation for the Galois action rather than the traditional contravariant one $f \mapsto f^\gamma$.

5.2 Open Questions

Theorem 24 shows the existence of a C^*-dynamical system $(\mathcal{A}_K, \sigma_t)$ with all the required properties for the interpretation of the class field theory isomorphism in the CM case in the framework of fabulous states. There is however still one key feature of the BC-system that needs to be obtained in this framework. It is the presentation of the arithmetic subalgebra $\mathcal{A}_{K,\mathbb{Q}}$ in terms of generators and relations. This should be obtained along the lines of [10] Section 6, Lemma 15 and Proposition 15, and Section 9 Proposition 41. These suggest that the relations will have coefficients in the Hilbert modular field.

We only handled in this paper the CM-case *i.e.* imaginary quadratic fields, but many of the notions we introduced such as that of a K-lattice should be extended to arbitrary number fields K. Note in that respect that Proposition 18 indicates clearly that, in general, the space of commensurability classes of K-lattices should be identical to the space $\mathbb{A}_K/K^*$ of Adèle classes introduced in [8] for the spectral realization of zeros of L-functions, with the slight nuance of non-zero archimedan component. The scaling group which is used to pass from the above "dual system" to the analogue of the BC system is given in the case $K = \mathbb{Q}$ by the group $\mathbb{R}_+^*$ and in the case of imaginary quadratic fields by the multiplicative group $\mathbb{C}^*$. It is thus natural to expect in general that it will be given by the connected component of identity D_K in the group C_K of idèle classes. A possible construction in terms of K-lattices should be compared with the systems for number fields recently constructed in [13].

There is another important conceptual point that deserves further investigation. In the classical class-field theory and Langlands program for a field K, the main object which is constructed is a certain "space", which is acted upon by two mutually commuting sets of operators: one is the Galois group of an extension of K, while the other is a group (algebra, semigroup, set) of geometric symmetries (like Hecke operators). In the very special case of abelian class field theory, it turns out that the commutator of each of these sets essentially coincides with it (or with an appropriate completion), so that these two groups can be naturally identified. The quantum statistical mechanical systems considered in this paper, as well as the mentioned generalizations, give new geometric objects, in the form of quantum statistical mechanical systems, supporting a similar type of geometric symmetries (*e.g.* groups of symmetries of adelic nature associated to Shimura varieties). The intertwining between geometric and Galois actions is provided by the extremal KMS states at zero temperature and by the presence of an arithmetic subalgebra. Even in the abelian cases, like the BC system and the CM system, it is important to maintain the conceptual difference between geometric and Galois actions. For instance, in the CM case the intertwining via the states, which provides the identification, relies essentially on the classical theory of complex multiplication, through the use of Shimura reciprocity. In the non-abelian setting, this distinction becomes essential. One can already see that clearly in the case of the GL_2-system, where for a generic set of extremal KMS states at zero temperature one still obtains a Galois interpretation of the geometric action (again by means of the classical theory of Shimura reciprocity and automorphisms of the modular field), while other phenomena appear for non-generic states, which reflect more clearly the difference between these two actions. For more general case associated to Shimura varieties, while the geometric action occurs naturally as in the GL_2-case and the number fields case, the Galois side is still not understood.

References

1. J. Arledge, M. Laca, I. Raeburn, *Semigroup crossed products and Hecke algebras arising from number fields*, Doc. Math. 2 (1997), 115–138 (electronic).
2. J.-B. Bost and A. Connes, *Produits Euleriens et facteurs de type III*, C. R. Acad. Sci. Paris Sér. I Math. Vol.315 (1992), no. 3, 279–284.
3. J.B. Bost, A. Connes, *Hecke algebras, Type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (New Series) Vol.1 (1995) N.3, 411–457.
4. O. Bratteli, D.W. Robinson, *Operator algebras and quantum statistical mechanics I,II*, Springer Verlag, 1981.
5. P.B. Cohen, *A C^*-dynamical system with Dedekind zeta partition function and spontaneous symmetry breaking*, Journal de Théorie des Nombres de Bordeaux 11 (1999) 15–30.
6. A. Connes, *Une classification des facteurs de type III*, Ann. Sci. École Norm. Sup. (4) 6 (1973), 133-252.
7. A. Connes, *C^* algèbres et géométrie différentielle*. C. R. Acad. Sci. Paris Sér. A-B 290 (1980), no. 13, A599–A604.
8. A. Connes, *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function*, Selecta Math. (New Series) Vol.5 (1999) 29–106.
9. A. Connes, M. Douglas, A. Schwarz, *Noncommutative geometry and Matrix theory: compactification on tori*. J. High Energy Phys. (1998) no. 2, Paper 3, 35 pp. (electronic)
10. A. Connes, M. Marcolli, *From physics to number theory via noncommutative geometry. Part I: Quantum statistical mechanics of $\mathbb{Q}$-lattices*, preprint 2004.
11. A. Connes, M. Marcolli, N. Ramachandran, *KMS states and complex multiplication*, Selecta Math. (N.S.) Vol. 11 (2005) N.3
12. A. Connes, H. Moscovici, *Modular Hecke algebras and their Hopf symmetry*, preprint math.QA/0301089.
13. E. Ha, F. Paugam, *Bost-Connes-Marcolli systems for Shimura varieties. I. Definitions and formal analytic properties*, preprint math.OA/0507101 (to appear in IMRN)
14. R. Haag, *Local Quantum Physics*, Springer, Berlin 1992.
15. R. Haag, N. M. Hugenholtz, M. Winnink *On the equilibrium states in quantum statistical mechanics*, Comm. Math. Phys. 5 (1967), 215, 236.
16. D. Harari, E. Leichtnam, *Extension du phénomène de brisure spontanée de symétrie de Bost–Connes au cas des corps globaux quelconques*, Selecta Math. (New Ser.) 3 (1997) 205–243.
17. D. Kubert, S. Lang, *Modular units*, Springer 1981.
18. M. Laca, *Semigroups of $*$-endomorphisms, Dirichlet series, and phase transitions*, J. Funct. Anal. 152 (1998) 330–378.
19. M. Laca, *From endomorphisms to automorphisms and back: dilations and full corners*, J. London Math. Soc. (2) 61 (2000) 893–904.
20. M. Laca, M. van Frankenhuijsen, *Phase transitions on Hecke C^*-algebras and class-field theory over $\mathbb{Q}$*, preprint math.OA/0410305.
21. S. Lang, *Elliptic functions*, (Second Edition), Graduate Texts in Mathematics **112**, Springer-Verlag 1987.
22. Yu.I. Manin, *Real Multiplication and noncommutative geometry*, preprint arXiv math.AG/0202109.

23. Yu.I. Manin, *Von Zahlen und Figuren*, preprint arXiv math.AG/0201005.
24. Yu.I. Manin, M. Marcolli, *Continued fractions, modular symbols, and noncommutative geometry*, Selecta Math. (New Series) Vol.8 (2002) N.3, 475–520.
25. J.S. Milne, *Canonical models of Shimura curves*, manuscript, 2003 (www.jmilne.org).
26. J. Milne, *Fields and Galois theory*, manuscript, (www.jmilne.org).
27. J. Milnor, *Introduction to algebraic K-theory*, Ann. Math. Studies, Vol.72, Princeton University Press, 1971.
28. S. Neshveyev, *Ergodicity of the action of the positive rationals on the group of finite adeles and the Bost-Connes phase transition theorem*, Proc. Amer. Math. Soc. 130 (2002), no. 10, 2999–3003.
29. F. Paugam, *Three examples of noncommutative boundaries of Shimura varieties*, preprint math.AG/0410254.
30. W. Rudin, *Real and Complex Analysis*, McGraw-Hill, NewYork 1987.
31. B. Schoeneberg, *Elliptic Modular Functions*, Springer, 1974.
32. G. Shimura, *Arithmetic theory of automorphic functions*, Iwanami Shoten and Princeton 1971.
33. P. Stevenhagen, *Hilbert's 12th problem, complex multiplication and Shimura reciprocity*, Advanced Studies in Pure Math. 30 (2001) "Class Field Theory - its centenary and prospect" pp. 161–176.
34. A. Weil, *Elliptic functions according to Eisenstein and Kronecker*, Springer 1976.

An Algebraic Description of Boundary Maps Used in Index Theory

Joachim Cuntz*

Mathematisches Institut, Universität Münster, Einsteinstr. 62, 48149 Münster, cuntz@math.uni-muenster.de

Dedicated to the memory of Gert Pedersen

In index theory and in noncommutative geometry one often associates C^*-algebras with geometric objects. These algebras can for instance arise from pseudodifferential operators, differential forms, convolution algebras etc.. However they are often given a priori as locally convex algebras and one looses a certain amount of information by passing to the C^*-algebra completions. In some cases, for instance for algebras containing unbounded differential operators, there is in fact no C^*-algebra that accommodates them. On the other hand, it seems that nearly all algebraic structures arising from differential geometry can be described very naturally by locally convex algebras (or by the slightly more general concept of bornological algebras). The present note can be seen as part of a program in which we analyze constructions, that are classical in K-theory for C^*-algebras and in index theory, in the framework of locally convex algebras. Since locally convex algebras have, besides their algebraic structure, only very little structure, all arguments in the study of their K-theory or their cyclic homology have to be essentially algebraic (thus in particular they also apply to bornological algebras).

This paper is triggered by an analysis of the proof of the Baum-Douglas-Taylor index theorem, [2], in the locally convex setting. Consider the extension

$$\mathcal{E}_\Psi : \quad 0 \to \mathcal{K} \to \Psi(M) \to \mathcal{C}(S^*M) \to 0$$

determined by the C^*-algebra completion $\Psi(M)$ of the algebra of pseudodifferential operators of order 0 on M and the natural extension

$$\mathcal{E}_{B^*M} : \quad 0 \to \mathcal{C}_0(T^*M) \to \mathcal{C}(B^*M) \to \mathcal{C}(S^*M) \to 0$$

determined by the evaluation map on the boundary S^*M of the ball bundle B^*M. Both extensions determine elements which we denote by $KK(\mathcal{E}_\Psi)$ and $KK(\mathcal{E}_{B^*M})$, respectively, in the bivariant K-theory of Kasparov.

*Research supported by the Deutsche Forschungsgemeinschaft

The Baum-Douglas-Taylor index theorem determines the K-homology class $KK(\mathcal{E}_\Psi)$ in $KK_1(\mathcal{C}(S^*M), \mathbb{C})$ as

$$KK(\mathcal{E}_\Psi) = KK(\mathcal{E}_{B^*M}) \cdot [\bar{\partial}_{T^*M}]$$

where $[\bar{\partial}_{T^*M}]$ is the fundamental K-homology class defined by the Dolbeault operator on T^*M. Note that multiplication by $KK(\mathcal{E}_{B^*M})$ describes the boundary map $K^0(\mathcal{C}_0(T^*M)) \to K^1(\mathcal{C}(S^*M)$ in K-homology.

This theorem which determines $KK(\mathcal{E}_\Psi)$ may be considered as a fundamental theorem in index theory, since it contains all the relevant information on the K-theoretic connections between symbols and indices of pseudodifferential operators on a given manifold. In particular, it contains the classical Atiyah-Singer theorem as well as Kasparov's bivariant version of the index theorem and determines not only the index of a given elliptic operator P, but also the K-homology class determined by P. (Note however that Kasparov proves his theorem in the equivariant case and for manifolds which are not necessarily compact. In this generality Kasparov's theorem remains the strongest result). The connection between the index theorems by Baum-Douglas-Taylor, Kasparov and Atiyah-Singer will be explained briefly in section 6.

The proof, by Baum-Douglas-Taylor, of their index theorem is a combination of a formula by Baum and Douglas [1], with a construction of Boutet de Monvel [4], [3]. The formula of Baum-Douglas determines the image under the boundary map, in the long exact sequence associated with an extension, of K-homology elements described by cycles satisfying certain conditions.

From this formula they derive a formula for the bivariant K-theory class determined by the Toeplitz extension on a strictly pseudoconvex domain. The construction by Boutet de Monvel identifies the extension of pseudodifferential operators on a manifold M with the Toeplitz extension on the strictly pseudoconvex domain given by the ball bundle on M with boundary given by the sphere bundle.

The original proof by Baum-Douglas of their formula for the boundary map has been streamlined substantially by Higson in [10], [11]. Higson gives in fact two proofs. One makes use of Skandalis' connection formalism the other one of Paschke duality, see also [12]. Both approaches rely on a certain amount of technical background. Higson also proves a formula in the case of odd Kasparov modules (Baum-Douglas consider only the even case). We also mention, even though this is not of direct relevance to our purposes that a simplification of the proof that the relative K-homology of Baum-Douglas coincides with the K-homology of the ideal is due to Kasparov [13].

We give algebraic proofs for the boundary map formula in the even and in the odd case. It turns out that the Baum-Douglas situation is exactly the one where the cycle representing the given K-homology element extends to a cycle for a K-homology element of the mapping cone or dual mapping cone (in the sense of [5]) associated with the given extension, respectively. This leads to a simple proof in the odd case. In the even case there is a completely direct proof which is very short. This proof depends on a new description of the boundary map which uses comparison to a free extension. We also include another, slightly longer proof, using the dual mapping cones of [5], because of its complete parallelism to the proof in the odd case using the ordinary mapping cone. The dual mapping cones have also been used in the paper by Baum-Douglas and this second proof resembles the proof by Baum-Douglas, but it has been reduced to its algebraic content.

Our discussion contains much more material than what is needed to determine the boundary map in the Baum-Douglas situation. We give different descriptions of the boundary map for bivariant K-theory, but also for more general homotopy functors. The argument for the Baum-Douglas formula itself is very short indeed and essentially contained in 4.2. In other descriptions of the boundary map we also have to study its compatibility with the Bott isomorphism. This compatibility has some interest for its own sake.

We would also like to emphasize the fact that our argument, even though formulated in the category of locally convex algebras for convenience, is completely general. Because of its algebraic nature it works in many other categories of topological algebras. In particular it can be readily applied to the category of C^*-algebras and gives there the original result of Baum-Douglas. For this, one has to use the appropriate analogs of the tensor algebra and of its ideal JA in the category of C^*-algebras as explained in [7]. We will discuss this in section 5.

1 Boundary Maps and Bott Maps

1.1 Locally Convex Algebras

By a locally convex algebra we mean an algebra over $\mathbb{C}$ equipped with a complete locally convex topology such that the multiplication $A \times A \to A$ is (jointly) continuous. This means that, for every continuous seminorm α on A, there is another continuous seminorm α' such that

$$\alpha(xy) \leq \alpha'(x)\alpha'(y)$$

for all $x, y \in A$. Equivalently, the multiplication map induces a continuous linear map $A \hat{\otimes} A \to A$ from the completed projective tensor product $A \hat{\otimes} A$. All homomorphisms between locally convex algebras will be assumed to be continuous. Every Banach algebra or projective limit of Banach algebras obviously is a locally convex algebra. But so is every algebra over $\mathbb{C}$ with a countable basis if we equip it with the "fine" locally convex topology, see e.g. [8]. The fine topology on a complex vector space V is given by the family of *all* seminorms on V. Homomorphisms between locally convex algebras will always be assumed to be continuous. We denote the category of locally convex algebras by LCA.

1.2 The Boundary Map for Half-Exact Homotopy Functors

By a well-known construction any half-exact homotopy functor on a category of topological algebras associates with any extension a long exact sequence which is infinite to one side. This construction is in fact quite general and works for different notions of homotopy (continuous, differentiable or $\mathcal{C}^\infty$) and on different categories of algebras as well as for different notions of extensions.

We review this construction here in some detail, since we need, for our purposes, the explicit description of the boundary map in the long exact sequence. To be specific we will work in the category of locally convex algebras with $\mathcal{C}^\infty$-homotopy. An extension will be a sequence

$$0 \to I \to A \to B \to 0$$

of locally convex algebras, where the arrows are continuous homomorphisms, which is split exact in the category of locally convex vector spaces, i.e. for which there is a continuous linear splitting $s : B \to A$. An extension will be called a split-extension if there is a continuous splitting $B \to A$ which at the same time is a homomorphism.

Let $[a, b]$ be an interval in $\mathbb{R}$. We denote by $\mathbb{C}[a, b]$ the algebra of complex-valued $\mathcal{C}^\infty$-functions f on $[a, b]$, all of whose derivatives vanish in a and in b (while f itself may take arbitrary values in a and b). Also the subalgebras $\mathbb{C}(a, b], \mathbb{C}[a, b)$ and $\mathbb{C}(a, b)$ of $\mathbb{C}[a, b]$, which, by definition consist of functions f, that vanish in a, in b, or in a and b, respectively, will play an important role. The topology on these algebras is the usual Fréchet topology.

Given two complete locally convex spaces V and W, we denote by $V \hat{\otimes} W$ their completed projective tensor product (see [14], [8]). We note that $\mathbb{C}[a, b]$ is nuclear in the sense of Grothendieck [14] and that, for any complete locally convex space V, the space $\mathbb{C}[a, b]\hat{\otimes}V$ is isomorphic to the space of $\mathcal{C}^\infty$-functions on $[a, b]$ with values in V, whose derivatives vanish in both endpoints, [14], § 51.

Given a locally convex algebra A, we write $A[a, b]$, $A(a, b]$ and $A(a, b)$ for the locally convex algebras $A\hat{\otimes}\mathbb{C}[a, b]$, $A\hat{\otimes}\mathbb{C}(a, b]$ and $A\hat{\otimes}\mathbb{C}(a, b)$ (their elements are A - valued $\mathcal{C}^\infty$-functions whose derivatives vanish at the endpoints). The algebra $A(0, 1]$ is called the cone over A and denoted by CA. The algebra $A(0, 1)$ is called the suspension of A and denoted by SA. The cone extension for A is

$$0 \to SA \to CA \to A \to 0$$

(it has an obvious continuous linear splitting). This extension is fundamental for the construction of the boundary maps. In the following we will usually consider covariant functors E on the category LCA. The contravariant case is of course completely analogous and, in fact, in later sections we will also apply the results discussed here to contravariant functors (such as K-homology).

Definition 1. *Let* E : LCA $\to$ Ab *be a functor from the category of locally convex algebras to the category of abelian groups. We say that*

- *E is diffotopy invariant, if the maps $\mathrm{ev}_t : E(A[0,1]) \to E(A)$ induced by the different evaluation maps for $t \in [0,1]$ are all the same (it is easy to see that this is the case if and only if the map induced by evaluation at $t = 0$ is an isomorphism).*
- *E is half-exact, if, for every extension $0 \to I \to A \to B \to 0$ of locally convex algebras, the induced short sequence $E(I) \to E(A) \to E(B)$ is exact.*

Definition 2. *Two homomorphisms $\alpha, \beta : A \to B$ between locally convex algebras are called diffotopic if there is a homomorphism $\varphi : A \to B[0,1]$ such that*

$$\alpha = \mathrm{ev}_0 \circ \varphi \qquad \beta = \mathrm{ev}_1 \circ \varphi.$$

A locally convex algebra A is called contractible if the endomorphisms id_A and 0 are diffotopic. If α and β are diffotopic and E is diffotopy invariant, then clearly $E(\alpha) = E(\beta)$. Moreover $E(A) = 0$ for every contractible algebra A.

Let $\alpha : A \to B$ be a continuous homomorphism between locally convex algebras. The mapping cone $C_\alpha \subset A \oplus B(0,1]$ is defined to be

$$C_\alpha = \{(x,f) \in A \oplus B(0,1] \,|\, \alpha(x) = f(1)\}$$

Similarly, the mapping cylinder Z_α is

$$Z_\alpha = \{(x,f) \in A \oplus B[0,1] \,|\, \alpha(x) = f(1)\}$$

Lemma 3. *(a) The maps $Z_\alpha \to A$, $(x,f) \mapsto x$ and $A \to Z_\alpha$, $x \mapsto (x, \alpha(x)1)$ are homotopy inverse to each other, i.e. their compositions both ways are diffotopic to the identity on Z_α and on A, respectively.*

(b) If there is a continuous linear map $s : B \to A$ such that $\alpha \circ s = \mathrm{id}_B$, then the natural exact sequence

$$0 \to C_\alpha \to Z_\alpha \to B \to 0$$

is an extension (i.e. admits a continuous linear splitting).

(c) If E is half-exact and $\pi : C_\alpha \to A$ is defined by $\pi((x,f)) = x$, then the sequence

$$E(C_\alpha) \xrightarrow{E(\pi)} E(A) \xrightarrow{E(\alpha)} E(B)$$

is exact.

Lemma 4. *Let $0 \to I \to A \xrightarrow{q} B \to 0$ be an extension of locally convex algebras. Denote by $e : I \to C_q$ the map defined by $e(x) = (x, 0) \in C_q \subset A \oplus CB$.*

(a) The following diagram commutes

$$\begin{array}{ccccccc} & & I & \to & A & \to & B \\ & & \downarrow e & & \| & & \downarrow \\ 0 \to SB & \xrightarrow{\kappa} & C_q & \xrightarrow{\pi} & A & \to & 0 \end{array}$$

and the natural map $\kappa : SB \to C_q$ *defined by* $\kappa(f) = (0, f)$ *makes the second row exact.*

(b) One has $E(C_e) = 0$ *and the map* $E(e) : E(I) \to E(C_q)$ *is an isomorphism.*

Proof. (a) Obvious. (b) This follows from the exact sequences $0 = E(CI) \to E(C_e) \to E(SCB) = 0$ and $E(C_e) \to E(I) \to E(C_q)$ (cf. 3 (c)).

Proposition 5. *Let*

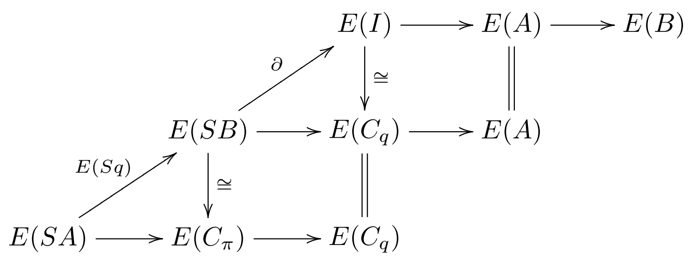
$$0 \to I \xrightarrow{j} A \xrightarrow{q} B \to 0$$

be an extension of locally convex algebras. Then there is a long exact sequence

$$\begin{array}{l} \xrightarrow{\partial} E(SI) \xrightarrow{E(Sj)} E(SA) \xrightarrow{E(S\alpha)} E(SB) \\ \qquad\qquad \xrightarrow{\partial} \;\; E(I) \;\; \xrightarrow{E(j)} \;\; E(A) \xrightarrow{E(q)} E(B) \end{array}$$

which is infinite to the left. The boundary map ∂ *is given by* $\partial = E(e)^{-1}E(\kappa)$.

Proof. Let $\pi : C_q \to A$ be as above. Consider the following diagram



The rows are exact and the diagram is commutative except possibly for the first triangle. By a well known argument one shows that the composition of the maps $SA \xrightarrow{Sq} SB \to C_\pi$ is diffotopic to the natural map $SA \to C_\pi$ composed with the self-map of SA that switches the orientation of the interval $[0, 1]$.

In the category of locally convex algebras we can also define an algebraic suspension and algebraic mapping cones in the following way.

Definition 6. *Let* $A[t] = A \hat{\otimes} \mathbb{C}[t]$ *denote the algebra of polynomials with coefficients in* A. *The topology is defined by choosing the fine topology on* $\mathbb{C}[t]$. *We denote by* $C^{\text{alg}}A$ *and* $S^{\text{alg}}A$ *the ideals* $tA[t]$ *and* $t(1-t)A[t]$ *of polynomials vanishing in 0 or in 0 and 1, respectively.*

Clearly, $C^{\text{alg}}A$ is contractible and we have an extension $0 \to S^{\text{alg}}A \to C^{\text{alg}}A \to A \to 0$. The associated long exact sequence shows that $E(S^{\text{alg}}A) = E(SA)$ for every half-exact diffotopy functor E.

The algebraic mapping cone C_α^{alg} for a homomorphism $\alpha : A \to B$ is defined as the subalgebra of $A \oplus C^{\text{alg}}B$ consisting of all pairs (x, f) such that $\alpha(x) = f(1)$. Again it is easily checked that the natural map $E(C_\alpha^{\text{alg}}) \to E(C_\alpha)$ is an isomorphism for every half-exact diffotopy functor E (compare the long exact sequences associated with the extensions $0 \to S^{\text{alg}}B \to C_\alpha^{\text{alg}} \to A \to 0$ and $0 \to SB \to C_\alpha \to A \to 0$).

1.3 The Universal Boundary Map

A description of the boundary map which is, at the same time, elementary and universal can be obtained by comparing a given extension to a free extension.

Let V be a complete locally convex space. Consider the algebraic tensor algebra

$$T_{alg}V = V \oplus V \otimes V \oplus V^{\otimes^3} \oplus \dots$$

with the usual product given by concatenation of tensors. There is a canonical linear map $\sigma : V \to T_{alg}V$ mapping V into the first direct summand. We equip $T_{alg}V$ with the locally convex topology given by the family of all seminorms of the form $\alpha \circ \varphi$, where φ is any homomorphism from $T_{alg}V$ into a locally convex algebra B such that $\varphi \circ \sigma$ is continuous on V, and α is a continuous seminorm on B. We further denote by TV the completion of $T_{alg}V$ with respect to this locally convex structure. TV has the following universal property:

for every continuous linear map $s : V \to B$ where B is a locally convex algebra, there is a unique homomorphism $\tau_s : TV \to B$ such that $s = \varphi \circ \sigma$.

(Proof. τ_s maps $x_1 \otimes x_2 \otimes \dots \otimes x_n$ to $s(x_1)s(x_2)\dots s(x_n) \in B$.)

For any locally convex algebra A we have the natural extension

$$0 \to JA \to TA \xrightarrow{\pi} A \to 0.$$

Here π maps a tensor $x_1 \otimes x_2 \otimes \dots \otimes x_n$ to $x_1 x_2 \dots x_n \in A$ and JA is defined as $\operatorname{Ker}\pi$. This extension is (uni)versal in the sense that, given any extension $\mathcal{E} : 0 \to I \to D \to B \to 0$ of a locally convex algebra B with continuous linear splitting s, and any continuous homomorphism $\alpha : A \to B$, there is a morphism of extensions

$$\begin{array}{ccccccc} 0 \to & JA & \to & TA & \to & A & \to 0 \\ & \downarrow \gamma & & \downarrow \tau & & \downarrow \alpha & \\ 0 \to & I & \to & D & \to & B & \to 0 \end{array}$$

The map $\tau : TA \to D$ maps $x_1 \otimes x_2 \otimes \dots \otimes x_n$ to $s'(x_1)s'(x_2)\dots s'(x_n) \in D$, where $s' := s \circ \alpha$.

Definition 7. *If $\alpha = \mathrm{id} : B \to B$, then the map $\gamma : JB \to I$ (defined to be the restriction of τ) is called the* classifying *map for the extension $\mathcal{E}$.*

The classifying map depends on s only up to diffotopy. In fact, if $\bar{s}$ is a second continuous linear splitting, then the classifying maps associated to $ts+(1-t)\bar{s}$ define a diffotopy between γ and the classifying map associated with $\bar{s}$. Thus, up to diffotopy, an extension has a unique classifying map. More generally, let $s : A \to D$ be a continuous linear map between locally convex algebras and I a closed ideal in D such that $s(xy) - s(x)s(y)$ is in I for all $x, y \in A$. Then the restriction γ_s, of τ_s to JA, maps JA into I. We have the following useful observation.

Lemma 8. *If $s' : A \to D$ is a second continuous linear map which is congruent to s in the sense that $s(x) - s'(x) \in I$ for all $x \in A$, then $\gamma_s, \gamma_{s'} : JA \to I$ are diffotopic.*

Proof. The diffotopy is induced by the linear map $\hat{s} : A \to D[0,1]$, where $\hat{s}_t = ts + (1-t)s'$.

Denote the classifying map $JB \to SB$ for the cone extension $0 \to SB \to CB \xrightarrow{p} B \to 0$ by ψ_B. Let E be a half-exact diffotopy functor. Comparing the long exact sequneces for the extension $0 \to JB \to TB \to B \to 0$ and for the cone extension $0 \to SB \to CB \to B \to 0$ gives

$$\begin{array}{ccccccc}
E(SB) & \xrightarrow{\partial_B} & E(JB) & \longrightarrow & E(TB) & \longrightarrow & E(B) \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle E(\psi_B)} & & \downarrow & & \downarrow \\
E(SB) & \xrightarrow{=} & E(SB) & \longrightarrow & E(CB) & \longrightarrow & E(B)
\end{array}$$

Moreover $E(CB) = E(TB) = 0$, since CB and TB are contractible. Therefore the boundary map, which we denote here by ∂_B, is an isomorphism and $\partial_B = E(\psi_B)^{-1}$.

Proposition 9. *Let $0 \to I \to A \to B \to 0$ be an extension of locally convex algebras with classifying map $\gamma : JB \to I$. Then the boundary map $\partial : E(SB) \to E(I)$ in the long exact sequence associated with this extension is given by the formula $\partial = E(\gamma) \circ \partial_B = E(\gamma) \circ E(\psi_B)^{-1}$.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & I & \longrightarrow & A & \longrightarrow & B & & \\
 & & \downarrow{\scriptstyle e} & & \downarrow & & \| & & \\
0 & \longrightarrow & C_q & \xrightarrow{\pi} & Z_q & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \kappa\circ\psi_B} & & \uparrow & & \| & & \\
 & & JB & \longrightarrow & TB & \longrightarrow & B & &
\end{array}$$

The maps $e \circ \gamma$ and $\kappa \circ \psi_B$ are both classifying maps for the extension in the middle row. By uniqueness of the classifying map they are diffotopic. Therefore $\partial = E(e)^{-1}E(\kappa) = E(e)^{-1}E(e)E(\gamma)E(\psi_B)^{-1}$.

1.4 The Toeplitz Extension and Bott Periodicity

The algebraic Toeplitz algebra $\mathcal{T}^{\mathrm{alg}}$ is the unital complex algebra with two generators v and v^* satisfying the identity $v^*v = 1$. It is a locally convex algebra with the fine topology. There is a natural homomorphism $\mathcal{T}^{\text{alg}} \to \mathbb{C}[z, z^{-1}]$ to the algebra of Laurent polynomials. The kernel is isomorphic to the algebra

$$M_\infty(\mathbb{C}) = \varinjlim_k M_k(\mathbb{C})$$

of matrices of arbitrary size (to see this note that the kernel is the ideal generated by the idempotent $e = 1 - vv^*$. The isomorphism maps an element $v^n e (v^*)^n$ of the kernel to the matrix unit E_{nm} in $M_\infty(\mathbb{C})$). $M_\infty(\mathbb{C})$ is a locally convex algebra with the fine topology (which is also the inductive limit topology in the representation as an inductive limit).

Given a locally convex algebra A, we consider also the algebra $M_\infty A$ defined by

$$M_\infty(A) = M_\infty(\mathbb{C}) \hat{\otimes} A \cong \varinjlim_k M_k(A)$$

Another standard locally convex algebra is the algebra $\mathcal{K}$ of "smooth compact operators" consisting of all $\mathbb{N} \times \mathbb{N}$-matrices (a_{ij}) with rapidly decreasing matrix elements $a_{ij} \in \mathbb{C}$, $i, j = 0, 1, 2 \ldots$. The topology on $\mathcal{K}$ is given by the family of norms p_n, $n = 0, 1, 2 \ldots$, which are defined by

$$p_n\big((a_{ij})\big) = \sum_{i,j} |1+i|^n |1+j|^n \, |a_{ij}|$$

Thus, $\mathcal{K}$ is isomorphic to the projective tensor product $s \hat{\otimes} s$, where s denotes the space of rapidly decreasing sequences $a = (a_i)_{i \in \mathbb{N}}$.

Definition 10. *A functor* $E : \mathrm{LCA} \to \mathrm{Ab}$ *is called* M_∞*-stable* ($\mathcal{K}$*-stable*)*, if the natural inclusion* $A \to M_\infty A$ ($A \to \mathcal{K} \hat{\otimes} A$) *induces an isomorphism* $E(A) \to E(M_\infty A)$ ($E(A) \to E(\mathcal{K} \hat{\otimes} A)$) *for each locally convex algebra* A.

We introduce the dual suspension $\hat{S}A$ of a locally convex algebra A as the kernel of the natural map $A[z, z^{-1}] \to A$, that maps z to 1 and abbreviate $\hat{S}\mathbb{C}$ to $\hat{S}$. The dual cone $\hat{C}A$ is defined as the kernel of the canonical homomorphism $\mathcal{T}^{\text{alg}} \hat{\otimes} A \to A$ that maps v to 1 and $\hat{C}\mathbb{C}$ is abbreviated to $\hat{C}$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{S}A & \longrightarrow & A[z, z^{-1}] & \longrightarrow & A & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \hat{C}A & \longrightarrow & \mathcal{T}^{\text{alg}} \hat{\otimes} A & \longrightarrow & A & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M_\infty A & \longrightarrow & M_\infty A & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

The terminology "dual cone" and "dual suspension" is motivated by the following

Proposition 11. *For every half-exact and M_∞-stable diffotopy functor E and for every locally convex algebra A , we have $E(\hat{C}A) = 0$.*

Proof (Sketch of proof). In the terminology explained in 3.3 there is a quasi-homomorphism $(\varphi, \bar{\varphi}) : \hat{C}A \to M_\infty \hat{C}A[0,1]$ such that $E(\varphi_0, \bar{\varphi}_0) = E(j)$ while $E(\varphi_1, \bar{\varphi}_1) = 0$ (for a complete proof see [8], 8.1, 8.2).

We obtain the dual cone extension

$$0 \to M_\infty A \to \hat{C}A \to \hat{S}A \to 0$$

Applying the long exact sequence, one immediately gets

Proposition 12. *For every half-exact and M_∞-stable diffotopy functor E and for every locally convex algebra A there is a natural isomorphism β : $E(S\hat{S}A) \to E(A)$.*

Proof. $\beta : E(S\hat{S}A) \to E(M_\infty A) \cong E(A)$ is given by the boundary map in the long exact sequence for the dual cone extension.

Remark 13. It is clear that the dual suspension is very closely related to the construction of negative K-theory by Bass.

1.5 A Dual Boundary Map

In this subsection we assume throughout that E is a diffotopy invariant, half-exact and M_∞-stable functor. Let $\alpha : A \to B$ be a continuous homomorphism between locally convex algebras and $0 \to M_\infty B \to \hat{C}B \xrightarrow{\pi} \hat{S}B \to 0$ the dual cone extension for B. We define the dual mapping cone by

$$\hat{C}_\alpha = \{(x,y) \in \hat{S}A \oplus \hat{C}B \,|\, \hat{S}\alpha(x) = \pi(y)\}$$

There is a natural extension $0 \to M_\infty B \to \hat{C}_\alpha \to \hat{S}A \to 0$.

Consider now an extension $0 \to I \xrightarrow{j} A \to B \to 0$ and the dual mapping cone $\hat{C}_j$. There are two natural extensions

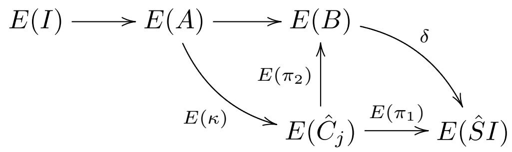
$$0 \to M_\infty A \to \hat{C}_j \xrightarrow{\pi_1} \hat{S}I \to 0$$

and

$$0 \to \hat{C}I \to \hat{C}_j \xrightarrow{\pi_2} M_\infty B \to 0$$

The second extension and the fact that $E(\hat{C}I) = 0$ for any locally convex algebra I, shows that $E(\pi_2) : E(\hat{C}_j) \to E(M_\infty B) \cong E(B)$ is an isomorphism.

Setting $\delta = E(\pi_1) \circ E(\pi_2)^{-1}$, we obtain the following commutative diagram



where $E(\kappa)$ is induced by the natural inclusion $\kappa : A \to M_\infty A \subset \hat{C}_j$.

Since $A \cong \kappa(A)$ is isomorphic under E to the kernel $M_\infty A$ of the natural surjection $\pi_1 : \hat{C}_j \to \hat{S}I$, it can be shown easily that the two sequences

$$E(I) \longrightarrow E(A) \longrightarrow E(B) \xrightarrow{\delta} E(\hat{S}I)$$

and

$$E(I) \longrightarrow E(A) \xrightarrow{\kappa} E(\hat{C}_j) \xrightarrow{E(\pi_1)} E(\hat{S}I)$$

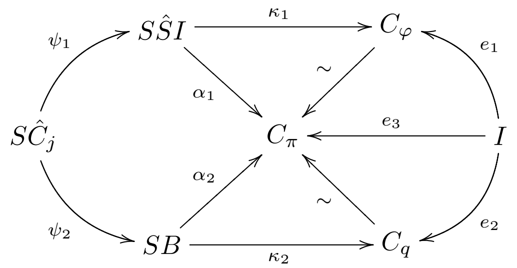
obtained from this diagram are exact and can in fact be continued indefinitely to the right (this is the dual mapping cone sequence discussed in [5]). We mention that the surjective map $\hat{C}_j \xrightarrow{\pi} \hat{S}I \oplus B$ with $\pi = \pi_1 \oplus \pi_2$ is exactly analogous to the dual inclusion map $SB \oplus I \longrightarrow C_q$ that has been used in the construction of the boundary map ∂ in 5.

We show now that, after identification by the Bott isomorphism β, the boundary maps ∂ and δ coincide up to a sign.

Proposition 14. *Let* $\partial : E(SB) \to E(I)$ *be the boundary map for the extension* $0 \to I \to A \to B \to 0$ *and* $\delta : E(SB) \to E(\hat{S}SI)$ *the dual boundary map for the suspended extension* $0 \to SI \to SA \to SB \to 0$*. Then* $\partial = -\beta \circ \delta$

Proof. Consider the mapping cones C_φ, C_π and C_q for the natural surjections $\varphi : \hat{C}I \to \hat{S}I$, $\pi : \hat{C}_j \to \hat{S}I \oplus B$ and $q : A \to B$ together with the associated inclusion maps in the Puppe sequence $S\hat{S}I \to C_\varphi$, $S\hat{S}I \to C_\pi$, $SB \to C_\pi$ and $SB \to C_q$. We identify $S\hat{S}I$ with $\hat{S}SI$.

We obtain the following diagram in which the upper half and the lower half commute:



We have $E(e_1)^{-1}E(\kappa_1) = \beta$ and $E(e_2)^{-1}E(\kappa_2) = \partial$.

Moreover, $E(\alpha_1 \circ \psi_1) + E(\alpha_2 \circ \psi_2) = 0$, since $\alpha_1 \circ \psi_1 + \alpha_2 \circ \psi_2$ is the composition of the maps $S\hat{C}_j \to S\hat{S}I \oplus SB \to C_\pi$ in the mapping cone sequence for π. Thus

$$0 = E(e_3)^{-1}(E(\alpha_1 \circ \psi_1) + E(\alpha_2 \circ \psi_2)) = \beta E(\psi_1) + \partial E(\psi_2)$$

Since, by definition $\delta = E(\psi_1)E(\psi_2)^{-1}$, the assertion follows.

2 The Categories kk^{alg} and kk

From now on we will describe our constructions in the category kk^{alg}. Since kk^{alg} acts on every diffotopy invariant, half-exact functor which is also $\mathcal{K}$-stable in the sense of 10, statements derived in kk^{alg} will pass to any functor with these properties. Some of the statements that we prove in the kk^{alg}-setting could also be proved for functors which are just M_∞-stable, rather than $\mathcal{K}$-stable. This slight loss of generality could easily be recovered by the interested reader wherever necessary. We mention at any rate that the arguments below depend in general only on some formal properties of the theory kk^{alg} and work just as well for other functors or bivariant theories satisfying the same conditions.

Explicitly, kk_*^{alg} is defined as

$$kk_n^{\mathrm{alg}}(A,\, B\,) = \varinjlim_k [J^{k-n}A,\, \mathcal{K}\hat{\otimes}S^kB\,]$$

where, given two locally convex algebras C and D, $[C,D]$ denotes the set of diffotopy classes of homomorphisms from C to D, see [8].

Here are some properties of kk^{alg} which are essential for our constructions:

- Every continuous homomorphism $\alpha : A \to B$ determines an element $kk(\alpha)$ in $kk_0^{\mathrm{alg}}(A,B)$. Given two homomorphisms α and β, we have $kk(\alpha \circ \beta) = kk(\beta)kk(\alpha)$.

- Every extension $\mathcal{E} : 0 \to I \xrightarrow{i} A \xrightarrow{q} B \to 0$ determines canonically an element $kk(\mathcal{E})$ in $kk^{\rm alg}_{-1}(B, I)$. The class of the cone extension

$$0 \to SA \to A(0,1] \to A \to 0$$

is the identity element in $kk^{\rm alg}_0(A, A) = kk^{\rm alg}_{-1}(A, SA)$. If

$$\begin{array}{ccccccc} (\mathcal{E}) : 0 \to & A_1 & \to A_2 & \to & A_3 & \to 0 \\ & \downarrow \alpha & \downarrow & & \downarrow \beta & \\ (\mathcal{E}') : 0 \to & B_1 & \to B_2 & \to & B_3 & \to 0 \end{array}$$

is a morphism of extensions (a commutative diagram where the rows are extensions), then $kk(\mathcal{E})kk(\alpha) = kk(\beta)kk(\mathcal{E}')$.

Moreover, for each fixed locally convex algebra D, the functor $A \mapsto kk^{\rm alg}_0(D, A)$ is covariant, half-exact, diffotopy invariant and M_∞-stable, while $A \mapsto kk^{\rm alg}_0$ (A, D) is a contravariant functor with the same properties. Thus both of these functors have long exact sequences where the boundary maps are given by the construction described in section 1.2. We refer to [8] for more details.

In [9] we considered the category $kk^{\mathcal{L}^p}_*$ defined by $kk^{\mathcal{L}^p}_*(A, B) = kk^{\rm alg}_*(A,$ $B \hat{\otimes} \mathcal{L}^p)$ where $\mathcal{L}^p$ denotes the Schatten ideal of p-summable operators for $1 \le p < \infty$. We showed that it follows from a result in [6] that $kk^{\mathcal{L}^p}_*$ does not depend on p. Let us denote $kk^{\mathcal{L}^p}_*$ by kk_* (this notation was not used in [9]). As shown in [9], a big advantage of the resulting theory kk_* is that its coefficient ring can be determined as $kk_0(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$ and $kk_1(\mathbb{C}, \mathbb{C}) = 0$. Otherwise, the theory kk_* has the same good formal properties as $kk^{\rm alg}_*$ and there is a natural functor from the category $kk^{\rm alg}_*$ to kk_*. Thus all identities proved in the category $kk^{\rm alg}_*$ carry over to kk_*.

3 Abstract Kasparov Modules

Baum and Douglas consider K-homology elements in $K^0(A)$ for a C^*-algebra A which are represented by an even Kasparov module. Such a Kasparov module consists of a pair (φ, F), where φ is a homomorphism from A into the algebra $\mathcal{L}(H)$ of bounded operators on a $\mathbb{Z}/2$-graded Hilbert space $H = H_+ \oplus H_-$ and F is a (self-adjoint) element of $\mathcal{L}(H)$ such that φ is even, F is odd, and such that for all $x \in A$ the following expressions lie in the algebra $\mathcal{K}(H)$ of compact operators on H

$$\varphi(x)(F - F^2), \quad [\varphi(x), F]$$

In the direct sum decomposition of H, F and φ correspond to matrices of the form

$$F = \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix} \qquad \varphi = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}$$

The corresponding K-homology element can be described by an associated quasihomomophism (see below).

Higson considers also the case of K-homology elements in $K^1(A)$ represented by an odd Kasparov module. Such a module consists again of a pair (φ, F), where φ is a homomorphism from A into the algebra $\mathcal{L}(H)$ of bounded operators on a Hilbert space H (which is this time trivially graded) and F is a (self-adjoint) element of $\mathcal{L}(H)$ such that for all $x \in A$ we have $\varphi(x)(F - F^2)$, $[\varphi(x), F] \in \mathcal{K}(H)$. In this case the K-homology element defined by (φ, F) is the one associated with the extension

$$0 \to \mathcal{K}(H) \to D \to A \to 0$$

where D is the subalgebra of $A \oplus \mathcal{L}(H)$ generated by products of $x \oplus \varphi(x)$, $x \in A$ together with elements of the algebra generated by $1 \oplus F$.

We will now describe kk^{alg}-elements associated with a Kasparov module in an abstract setting.

3.1 Morphism Extensions

Let A and B be locally convex algebras. A morphism extension from A to B will be a diagram of the form

$$\begin{array}{ccccccccc} & & & & & & A & & \\ & & & & & & \downarrow{\scriptstyle\varphi} & & \\ 0 & \longrightarrow & K & \longrightarrow & D & \xrightarrow{q} & B & \longrightarrow & 0 \end{array}$$

where φ is a homomorphism and the row is an extension.

We can encode the information contained in a morphism extension in a single (pull back) extension in the following way.

Define D' as the subalgebra of $A \oplus D$ consisting of all elements (a, d) such that $\varphi(x) = q(d)$. The natural homomorphism $\pi : D' \to D$ defined by $\pi((a, d)) = d$ gives the following morphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & D' & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\varphi} & & \\ 0 & \longrightarrow & K & \longrightarrow & D & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

If $\mathcal{E}$ is the original extension in the second row and $\mathcal{E}'$ the pull back extension, then $kk(\mathcal{E}') = kk(\varphi)kk(\mathcal{E})$. We say that this element $kk(\mathcal{E}') = kk(\varphi)kk(\mathcal{E})$ is the element of $kk^{\mathrm{alg}}_{-1}(A, K)$ associated with the given morphism extension and denote it by $kk(\mathcal{E}, \varphi)$.

3.2 Abstract Odd Kasparov Modules

Definition 15. *Let A be a locally convex algebra and $0 \to K \to D \to D/K \to 0$ an extension of locally convex algebras where D is unital. An abstract odd Kasparov (A, K)-module relative to D is a pair (φ, P) where*

- *φ is a continuous homomorphism from A into D.*
- *P is an element in D such that the following expressions are in K for all $x \in A$:*

$$[P, \varphi(x)], \quad \varphi(x)(P - P^2)$$

With an odd Kasparov module we can associate the following morphism extension

$$\begin{array}{ccccccccc} & & & & & & A & & \\ & & & & & & \downarrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & K & \longrightarrow & D & \xrightarrow{q} & D/K & \longrightarrow & 0 \end{array}$$

where q is the quotient map and $\tau(x) = q(P\varphi(x)P)$. We denote by $kk(\varphi, P)$ the element of $kk^{\mathrm{alg}}_{-1}(A, K)$ associated with this morphism extension.

3.3 Quasihomomorphisms

Let α and $\bar{\alpha}$ be two homomorphisms $A \to D$ between locally convex algebras. Assume that B is a closed subalgebra of D such that $\alpha(x) - \bar{\alpha}(x) \in B$ and $\alpha(x)B \subset B$, $B\alpha(x) \subset B$ for all $x \in A$. We call such a pair $(\alpha, \bar{\alpha})$ a quasihomomorphism from A to B relative to D and denote it by $(\alpha, \bar{\alpha}) : A \to B$.

We will show that $(\alpha, \bar{\alpha})$ induces a homomorphism $E(\alpha, \bar{\alpha}) : E(A) \to E(B)$ in the following way. Define $\alpha', \bar{\alpha}' : A \to A \oplus D$ by $\alpha'(x) = (x, \alpha(x))$, $\bar{\alpha}' = (x, \bar{\alpha}(x))$ and denote by D' the subalgebra of $D \oplus A$ generated by all elements $\alpha'(x),\ x \in A$ and by $0 \oplus B$. We obtain an extension with two splitting homomorphisms α' and $\bar{\alpha}'$:

$$0 \to B \to D' \to A \to 0$$

where the map $D' \to A$ by definition maps $(x, \alpha(x))$ to x and $(0, b)$ to 0. The map $E(\alpha, \bar{\alpha})$ is defined to be $E(\alpha') - E(\bar{\alpha}') : E(A) \to E(B) \subset E(D')$ (this uses split-exactness). Note that $E(\alpha, \bar{\alpha})$ is independent of D in the sense that we can enlarge D without changing $E(\alpha, \bar{\alpha})$ as long as B maintains the properties above.

Proposition 16. *The assignment $(\alpha, \bar{\alpha}) \to E(\alpha, \bar{\alpha})$ has the following properties:*

(a) $E(\bar{\alpha}, \alpha) = -E(\alpha, \bar{\alpha})$
(b) If the linear map $\varphi = \alpha - \bar{\alpha}$ is a homomorphism and satisfies $\varphi(x)\bar{\alpha}(y) = \bar{\alpha}(x)\varphi(y) = 0$ for all $x, y \in A$, then $E(\alpha, \bar{\alpha}) = E(\varphi)$.

(c) Assume that α is diffotopic to α' and $\bar{\alpha}$ is diffotopic to $\bar{\alpha}'$ via diffotopies φ, $\bar{\varphi}$ such that $\varphi_t(x) - \alpha(x), \bar{\varphi}_t(x) - \bar{\alpha}(x) \in B$ for all $x \in A$ (we denote this situation by $\alpha \sim_B \alpha'$, $\bar{\alpha} \sim_B \bar{\alpha}'$). Then $E(\alpha, \bar{\alpha}) = E(\alpha', \bar{\alpha}')$.

Proof. (a) This is obvious from the definition. (b) This follows $\varphi + \bar{\alpha} = \alpha$ and the fact that $E(\varphi + \bar{\alpha}) = E(\varphi) + E(\bar{\alpha})$. (c) follows from the definition of $E(\alpha, \bar{\alpha})$ and diffotopy invariance of E.

Choosing $E(?) = kk_0^{\mathrm{alg}}(A, ?)$ produces in particular an element $kk(\alpha, \bar{\alpha})$ in $kk_0^{\mathrm{alg}}(A, B)$ (obtained by applying $E(\alpha, \bar{\alpha})$ to the unit element 1_A in $kk_0^{\mathrm{alg}}(A, A)$) .

3.4 Abstract Even Kasparov Modules

Definition 17. *Let A, K and D be locally convex algebras. Assume that D is unital and contains K as a closed ideal. An abstract even Kasparov (A, K)-module relative to D is a triple $(\alpha, \bar{\alpha}, U)$ where*

- *$\alpha, \bar{\alpha}$ are continuous homomorphisms from A into D.*
- *U is an invertible element in D such that $U\bar{\alpha}(x) - \alpha(x)U$ is in K for all $x \in A$.*

From an even Kasparov module we obtain a quasihomomorphism $(\alpha, \mathrm{Ad}\, U \circ \bar{\alpha}) : A \to K$. We write $kk(\alpha, \bar{\alpha}, U)$ for the element of $kk_0^{\mathrm{alg}}(A, K)$ associated with this quasihomomorphism. More generally, if E is a half-exact diffotopy functor we write $E(\alpha, \bar{\alpha}, U)$ for the morphism $E(A) \to E(K)$ obtained from this quasihomomorphism.

Remark 18. The connection with Kasparov's definition in the C^*-algebra/Hilbert space setting mentioned at the beginning of section 3 is obtained by setting

$$U = \begin{pmatrix} \sqrt{f_1} & v \\ -v^* & \sqrt{f_2} \end{pmatrix}$$

where $f_1 = 1 - vv^*$ and $f_2 = 1 - v^*v$, and by replacing $\alpha, \bar{\alpha}$ by $\alpha \oplus 0, 0 \oplus \bar{\alpha}$.

This corresponds to replacing the Kasparov module (H, F) by the inflated module (H', F') where H' is the $\mathbb{Z}/2$-graded Hilbert space $H \oplus H$ with $H = H_+ \oplus H_-$ and F by

$$F' = \begin{pmatrix} 0 & U \\ U^{-1} & 0 \end{pmatrix}$$

3.5 Special Abstract Even Kasparov Modules

Let $\varphi : A \to D$ be a homomorphism of locally convex algebras where D is unital and contains a closed ideal K. Assume that D contains elements v, v^*

such that the expressions

$$[\varphi(x), v], [\varphi(x), v^*], \varphi(x)(vv^* - 1), \varphi(x)(v^*v - 1)$$

are in K for all $x \in A$.

If we assume moreover that, in D, there are square roots for the elements $f_1 = 1 - vv^*$ and $f_2 = 1 - v^*v$, we can form an abstract even Kasparov module (relative to M_2D) by choosing

$$\alpha = \bar{\alpha} = \begin{pmatrix} \varphi & 0 \\ 0 & 0 \end{pmatrix} \qquad U = \begin{pmatrix} v & \sqrt{f_1} \\ \sqrt{f_2} & -v^* \end{pmatrix}$$

If we suppose in this case moreover that there is a continuous linear splitting $D/K \to D$, we can associate with this even Kasparov module (α, α, U) also a morphism extension

$$\begin{array}{ccccccccc} & & & & & & \hat{S}A & & \\ & & & & & & \downarrow{\scriptstyle \rho} & & \\ 0 & \longrightarrow & K & \longrightarrow & D & \xrightarrow{\pi} & D/K & \longrightarrow & 0 \end{array}$$

by defining $\rho(\sum x_i z^i) = \pi(\sum \alpha(x_i) U^i)$.

It can be checked that the element in $kk_{-1}^{\mathrm{alg}}(\hat{S}A, K)$ defined by this morphism extension corresponds to the element $kk(\alpha, \alpha, U)$ constructed above under the Bott isomorphism $kk_{-1}^{\mathrm{alg}}(\hat{S}A, K) \cong kk_0^{\mathrm{alg}}(A, K)$.

We will later consider the case where $v^*v = 1$ and thus $f_1 = 1 - vv^*$ is an idempotent.

Remark 19. In [9] we had considered a different notion of an even Kasparov module. This notion is closely related to the situation considered here.

4 The Boundary Map in the Baum-Douglas Situation

Baum and Douglas consider an extension $0 \to I \to A \to B \to 0$ of C^*-algebras and obtain for K-homology elements, that are realized by a special kind of Kasparov modules, a formula for the boundary map $K^0 I \to K^1 B$ in K-homology in the long exact sequence associated with the extension $0 \longrightarrow I \xrightarrow{j} A \xrightarrow{q} B \longrightarrow 0$. The basic assumption on the given element in $K^0 I$ for which the boundary is determined is that it should be realized by a Kasparov module (φ, F) for which φ extends to a homomorphism such that $\varphi(A)$ still commutes with F modulo compacts (however $\varphi(x)F^2 = \varphi(x)$ holds only for $x \in I$ and not necessarily for $x \in A$). Higson gives a similar formula under analogous conditions for the boundary map $K^1 I \to K^0 B$.

It turns out that the condition imposed by Baum-Douglas on the Kasparov module means exactly that the Kasparov module for $\hat{S}I$ extends to a Kasparov module for the dual mapping cone $\hat{C}_j$, while Higson's condition in the odd case means that the corresponding Kasparov module extends to one for the ordinary mapping cone C_q.

With this observation and the preliminaries explained in the previous sections it is completely straightforward to deduce explicit formulas for the images of the given K-homology elements under the boundary map.

4.1 The Odd Case

Higson has stated and proved a formula, for the image under the boundary map of certain odd K-homology elements, which is analogous to the Baum-Douglas formula for even elements, [10], [11]. In this subsection we give a simple proof for this formula. As in the even case, our proof carries over verbatim to the case of C^*-algebras and thus to the case considered by Higson. Let $0 \to I \to A \xrightarrow{q} B \to 0$ be an extension of locally convex algebras with a continuous linear splitting s. Assume, we are given an odd Kasparov (I, K)-module (φ, P), where φ is a continuous homomorphism into a locally convex algebra D, P is an element of D and K is a closed ideal in D. Thus by definition $[\varphi(I), P] \subset K$ and $\varphi(I)(P - P^2) \subset K$.

Suppose now that φ extends to a homomorphism $\varphi : A \to D$ such that also $[\varphi(A), P] \subset K$. Consider the morphism extension $(\mathcal{E}, \tau)$

$$\begin{array}{ccccccccc} & & & & & & I & & \\ & & & & & & \downarrow{\scriptstyle\tau} & & \\ 0 & \longrightarrow & K & \longrightarrow & D & \xrightarrow{\pi} & D/K & \longrightarrow & 0 \end{array}$$

associated to (φ, P) as in 3.2.

Proposition 20. *Let $\partial : kk^{\mathrm{alg}}_{-1}(I, K) \longrightarrow kk^{\mathrm{alg}}_{-1}(S^{\mathrm{alg}}B, K)$ be the boundary map. Then $\partial(kk(\mathcal{E}, \tau))$ is represented by the morphism extension $(\mathcal{E}, \psi)$ given by the diagram*

$$\begin{array}{ccccccccc} & & & & & & S^{\mathrm{alg}}B & & \\ & & & & & & \downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & K & \longrightarrow & D & \xrightarrow{\pi} & D/K & \longrightarrow & 0 \end{array}$$

where π is the quotient map and ψ is defined by $\psi(\sum b_i t^i) = \pi(\sum \varphi(sb_i)P^i)$.

Proof. We define a homomorphism $\rho : C^{\mathrm{alg}}_q \to D/K$ as follows. Let $(x, f) \in C^{\mathrm{alg}}_q$ where $f = \sum b_i t^i$ is in $C^{\mathrm{alg}}B$ and $f(1) = q(x)$. We set $\rho((x, f)) = \pi(\varphi(x - sq(x)) + \sum \varphi(sb_i)P^i)$.

We have now three morphism extensions defined by the extension $0 \to K \to D \to D/K \to 0$ together with the homomorphisms $I \to D/K$, $S^{\mathrm{alg}}B \to$

D/K and $C_q^{\mathrm{alg}} \to D/K$. Consider the three extensions $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ associated to these morphism extensions as in 3.1. We obtain the following commutative diagram of extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & D_1 & \longrightarrow & I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow e & & \\
0 & \longrightarrow & K & \longrightarrow & D_3 & \longrightarrow & C_q^{\mathrm{alg}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \kappa & & \\
0 & \longrightarrow & K & \longrightarrow & D_2 & \longrightarrow & S^{\mathrm{alg}}B & \longrightarrow & 0
\end{array}$$

where the first row is $\mathcal{E}_1$, the last one $\mathcal{E}_2$ and the extension in the middle is $\mathcal{E}_3$. It follows that $kk(\mathcal{E}_3)kk(\kappa) = kk(\mathcal{E}_2)$ and $kk(\mathcal{E}_3)kk(e) = kk(\mathcal{E}_1)$. Since $kk(e)$ is invertible we conclude $kk(\mathcal{E}_2) = kk(\mathcal{E}_1)kk(e)^{-1}kk(\kappa)$. But $kk(e)^{-1}kk(\kappa) = \partial$ by 5.

4.2 The Even Case

This subsection contains the proof of the Baum-Douglas formula for the image, under the boundary map, of certain even K-homology elements. Our proof is extremely short. It uses only a small part of the discussion above, namely the description of the boundary map in subsection 1.3.

Proposition 21. *Let $0 \to I \to A \to B \to 0$ be an extension of locally convex algebras and $s : B \to A$ a continuous linear splitting. Let $(\alpha, \bar{\alpha}, U)$ be an even (I, K)-Kasparov module relative to D and $z = kk(\alpha, \bar{\alpha}, U)$ the corresponding element of $kk_0^{\mathrm{alg}}(I, K)$. Then $\partial z \in kk_1^{\mathrm{alg}}(B, K) = kk_0^{\mathrm{alg}}(JB, K)$ is given by $kk((\alpha \oplus 0) \circ \gamma_s, \operatorname{Ad} U(\bar{\alpha} \oplus 0) \circ \gamma_s)$ where $\gamma_s : JB \to I$ is the classifying map.*

Proof. This follows immediately from 9 applied to $E(?) = kk_0^{\mathrm{alg}}(?, K)$ and using the identification $kk_0(JB, K) \cong kk_0(SB, K)$ via $E(\psi_B)$.

In order to transform this formula for ∂z into a more usable form we need the following trivial lemma.

Lemma 22. *Let B, D, K be locally convex algebras such that K is a closed ideal in D. Let $\rho, \bar{\rho} : B \to D$ be continuous linear maps such that for the induced maps $\gamma_\rho, \gamma_{\bar{\rho}} : JB \to D$ we have $\gamma_\rho(x) - \gamma_{\bar{\rho}}(x) \in K$ for all $x \in JB$. Assume moreover that $\rho', \bar{\rho}' : B \to D$ is another pair of continuous linear maps which are congruent to $\rho, \bar{\rho} : B \to D$ in the sense that $\rho(x) - \rho'(x) \in K$ and $\bar{\rho}(x) - \bar{\rho}'(x) \in K$ for all x in B. Then the quasihomomorphism $(\gamma_\rho, \gamma_{\bar{\rho}}) : JB \to K$ is diffotopic to $(\gamma'_\rho, \gamma'_{\bar{\rho}})$ in the sense of 16 (c).*

Proof. Let $\sigma, \bar{\sigma} : B \to D[0,1]$ denote the linear maps defined by $\sigma_t(x) = t\rho(x) + (1-t)\rho'(x)$ and $\bar{\sigma}_t(x) = t\bar{\rho}(x) + (1-t)\bar{\rho}'(x)$. Then $(\gamma_\sigma, \gamma_{\bar{\sigma}})$ defines a diffotopy between $(\gamma_\rho, \gamma_{\bar{\rho}})$ and $(\gamma'_\rho, \gamma'_{\bar{\rho}})$.

Consider again the situation of 21. We will assume now that the unital algebra D admits a "2×2-matrix decomposition" (i.e. D is a direct sum of subspaces D_{ij}, $i,j=1,2$, with $D_{ij}D_{jk}\subset D_{ik}$) and that $\alpha,\bar\alpha$ and U are of the form

$$\alpha(x)=\begin{pmatrix}\alpha_0(x) & 0\\ 0 & 0\end{pmatrix}\qquad \bar\alpha(x)=\begin{pmatrix}0 & 0\\ 0 & \bar\alpha_0(x)\end{pmatrix}\qquad U=\begin{pmatrix}e & v\\ -v^* & \bar e\end{pmatrix}$$

where $v\in D_{12}, v^*\in D_{21}$ are elements such that $e=1_{D_{11}}-vv^*$ and $\bar e=1_{D_{22}}-v^*v$ are idempotents. We also assume that the ideal K is compatible with this 2×2-matrix decomposition. Recall that this is a typical form in which Kasparov modules arise in applications.

Theorem 23. *Let $0\to I\to A\to B\to 0$ be an extension of locally convex algebras, $s:B\to A$ a continuous linear splitting and $\partial: kk_0^{\rm alg}(I,K)\to kk_{-1}^{\rm alg}(B,K)$ the associated boundary map. Let $(\alpha,\bar\alpha,U)$ as above represent an even (I,K)-module relative to D. Assume that $\alpha_0,\bar\alpha_0:I\to D$ extend to homomorphisms, still denoted by $\alpha_0,\bar\alpha_0$, from A to D_{11}, resp. to D_{22}, and assume moreover that the elements $v\bar\alpha_0(x)-\alpha_0(x)v$, $\bar\alpha_0(x)v^*-v^*\alpha_0(x)$ are in K for all $x\in A$. Let $z=kk(\alpha,\bar\alpha,U)\in kk_0^{\rm alg}(I,K)$. Then ∂z is represented by $kk(\gamma_\tau)-kk(\gamma_{\bar\tau})$ where $\tau,\bar\tau:B\to D$ are given by $\tau(x)=e\,\alpha_0 s(x)\,e$ and $\bar\tau(x)=\bar e\,\bar\alpha_0 s(x)\,\bar e$ and $\gamma_\tau,\gamma_{\bar\tau}:JB\to K$ are the corresponding homomorphisms.*

Proof. We have $\alpha\circ\gamma_s=\gamma_{\alpha\circ s}$, ${\rm Ad}U\circ\bar\alpha\circ\gamma_s=\gamma_{{\rm Ad}U\circ\bar\alpha\circ s}$ (here we use the fact that α and $\bar\alpha$ extend to A!). Therefore, from 21, the element ∂z is represented by the quasihomomorphism $(\gamma_\rho,\gamma_{\bar\rho}):JB\to K$ where $\rho(x)=\alpha(sx)$ and $\bar\rho(x)=U\bar\alpha(sx)U^{-1}$.
Writing $e^\perp=1-e=vv^*$ we have $v\bar\alpha_0(sx)v^*-e^\perp\alpha_0(sx)e^\perp\in K$ and $\alpha_0(sx)-e^\perp\alpha_0(sx)e^\perp-\tau(x)\in K$ for all $x\in B$. Therefore, setting $\rho_0(x)=\alpha_0(sx)$, $\bar\rho_0(x)=\bar\alpha_0(sx)$ we have the following congruences

$$\rho\simeq\begin{pmatrix}e^\perp\rho_0e^\perp+\tau & 0\\ 0 & 0\end{pmatrix}\qquad \bar\rho\simeq\begin{pmatrix}v\bar\rho_0v^* & 0\\ 0 & \bar\tau\end{pmatrix}$$

Thus, by Lemma 22,

$$\gamma_\rho\sim\begin{pmatrix}\gamma_{v\rho_0v^*}+\gamma_\tau & 0\\ 0 & 0\end{pmatrix}\qquad \gamma_{\bar\rho}\sim\begin{pmatrix}\gamma_{v\rho_0v^*} & 0\\ 0 & \gamma_{\bar\tau}\end{pmatrix}$$

whence $kk(\gamma_\rho,\gamma_{\bar\rho})=kk(\gamma_\tau,\gamma_{\bar\tau})=kk(\gamma_\tau)-kk(\gamma_{\bar\tau})$ (here it is important that the homomorphisms $\gamma_\tau,\gamma_{\bar\tau}$ themselves and not only their difference map into K).

4.3 The Dual Boundary Map in the Even Case

There is an alternative way of describing the boundary map in the Baum-Douglas situation using the dual mapping cone construction described in 1.5.

We limit our discussion here to the case of special (see 3.5) even Kasparov modules.

Let $0 \to I \xrightarrow{j} A \to B \to 0$ be an extension of locally convex algebras with a continuous linear splitting s. Assume that $\varphi : I \to D$ and $v, v^* \in D$ satisfy the conditions in 3.5 and thus, by the construction in 3.5, define an even Kasparov (I, K)-module relative to D. We now assume that it satisfies the Baum-Douglas condition that φ extends from I to a homomorphism still denoted φ from A to D such that $[\varphi(A), v], [\varphi(A), v^*] \subset K$. Assume moreover that $v^*v = 1$ and denote by e the idempotent $e = 1 - vv^*$.

Consider the morphism extension $(\mathcal{E}, \tau)$

$$\begin{array}{ccccccccc} & & & & & & \hat{S}I & & \\ & & & & & & \big\downarrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & K & \longrightarrow & D & \xrightarrow{\pi} & D/K & \longrightarrow & 0 \end{array}$$

associated to (φ, v) as in 3.5.

Proposition 24. *Let $\partial : kk^{\mathrm{alg}}_{-1}(\hat{S}I, K) \longrightarrow kk^{\mathrm{alg}}_{-1}(B, K)$ be the boundary map (where we identify $kk^{\mathrm{alg}}_{-1}(S\hat{S}B, K)$ with $kk^{\mathrm{alg}}_{-1}(B, K)$ via Bott periodicity). Then $\partial(kk(\mathcal{E}, \tau))$ is represented by the morphism extension $(\mathcal{E}, \psi)$ given by the diagram*

$$\begin{array}{ccccccccc} & & & & & & B & & \\ & & & & & & \big\downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & K & \longrightarrow & D & \xrightarrow{\pi} & D/K & \longrightarrow & 0 \end{array}$$

where π is the quotient map and ψ is defined by $\psi(b) = \pi(e\varphi(sb)e)$.

Proof. The proof is exactly analogous to the proof of 20. We define a homomorphism $\rho : \hat{C}_j \to D/K$ as follows. Let $f \in \hat{C}_j \subset \hat{C}A$ where $f = \sum a_i z^i$. We set $\rho(f) = \pi(\sum \varphi(a_i) v^i)$ (where we use the convention that $v^{-k} = (v^*)^k$ for $k \in \mathbb{N}$).

We have now three morphism extensions defined by the extension $0 \to K \to D \to D/K \to 0$ together with the homomorphisms $\hat{S}I \to D/K$, $B \to D/K$ and $\hat{C}_j \to D/K$. Consider now the three pull back extensions $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ associated to these morphism extensions as in 3.1. We obtain the following commutative diagram of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & D_1 & \longrightarrow & \hat{S}I & \longrightarrow & 0 \\ & & \big\uparrow & & \big\uparrow & & \big\uparrow{\scriptstyle \pi_1} & & \\ 0 & \longrightarrow & K & \longrightarrow & D_3 & \longrightarrow & \hat{C}_j & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \pi_2} & & \\ 0 & \longrightarrow & K & \longrightarrow & D_2 & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

where the first row is $\mathcal{E}_1$, the last one $\mathcal{E}_2$ and the extension in the middle is $\mathcal{E}_3$. It follows that $kk(\mathcal{E}_3) = kk(\pi_1)kk(\mathcal{E}_1)$ and $kk(\mathcal{E}_3) = kk(\pi_2)kk(\mathcal{E}_2)$. Since $kk(\pi_2)$ is invertible we conclude $kk(\mathcal{E}_2) = kk(\pi_2)^{-1}kk(\pi_1)kk(\mathcal{E}_1)$. But $kk(\pi_2)^{-1}kk(\pi_1) = \delta$ and δ corresponds to ∂ after identifying by Bott periodicity, see 14.

5 The Case of C^*-Algebras

We can carry through the construction of bivariant K-theory described above in the case of locally convex algebras in many other categories of algebras and in particular in the category of C^*-algebras. We have to replace the basic ingredients by the appropriate constructions in that category. Thus we replace:

the algebras of functions such as $A[a, b], A(a, b), SA, CA$ by the corresponding algebras of continuous (rather than smooth) functions, and diffotopy by homotopy
the locally convex algebra $\mathcal{K}$ of smooth compact operators by the C^*-algebra $\mathcal{K}$ of compact operators
the projective tensor product by the C^*-tensor product
the smooth Toeplitz algebra by the well known Toeplitz C^*-algebra
and - most importantly - the tensor algebra TA by the tensor algebra in the category of C^*-algebras described in the next paragraph.

Let A be a C^*-algebra. To construct the tensor algebra TA in the category of C^*-algebras consider as before the algebraic tensor algebra

$$T_{alg}A = A \oplus A \otimes A \oplus A^{\otimes^3} \oplus \dots$$

with product given by concatenation of tensors and let σ denote the canonical linear map $\sigma : A \to T_{alg}A$.

Equip $T_{alg}A$ with the C^*-norm given as the sup over all C^*-seminorms of the form $\alpha \circ \varphi$, where φ is any homomorphism from $T_{alg}A$ into a C^*-algebra B such that $\varphi \circ \sigma$ is completely positive contractive on A, and α is the C^*-norm on B. Let TA be the completion of $T_{alg}A$ with respect to this C^*-norm. TA has the following universal property:

for every contractive completely positive map $s : A \to B$ where B is a C^*-algebra, there is a unique homomorphism $\varphi : TA \to B$ such that $s = \varphi \circ \sigma$.

The tensor algebra extension:

$$0 \to JA \to TA \xrightarrow{\pi} A \to 0$$

is (uni)versal in the sense that, given any extension $0 \to I \to E \to B \to 0$ admitting a completely positive splitting, and any continuous homomorphism

$\alpha : A \to B$, there is a morphism of extensions

$$\begin{array}{ccccccccc} 0 \to & JA & \to & TA & \to & A & \to 0 \\ & \downarrow \gamma & & \downarrow \tau & & \downarrow \alpha & \\ 0 \to & I & \to & E & \to & B & \to 0 \end{array}$$

We can now define

$$KK_n(A,\, B\,) = \varinjlim_k [J^{k-n}A,\, \mathcal{K} \otimes S^k B\,]$$

This definition is exactly analogous to the definition of kk^{alg} in section 2. By the same arguments as in the case of kk^{alg} it is seen that this functor has a product, long exact sequences associated to extensions with cp splitting and is Bott-periodic and $\mathcal{K}$-stable. In fact, it is universal with these properties and this shows that KK gives an alternative construction for Kasparov's KK-functor.

The description of the boundary maps given above now carry over basically word by word. In particular, for an extension

$$0 \to I \to A \to B \to 0$$

of C^*-algebras with a completely positive splitting, the associated element in $KK_{-1}(B, I)$ is represented by the classifying map $\gamma_s : JB \to I$ and the boundary map is given by composition with this homomorphism.

6 The Index Theorem of Baum-Douglas-Taylor

For completeness we briefly recall the argument by Baum-Douglas-Taylor in [2].

Let M be a compact $\mathcal{C}^\infty$-manifold. Then a neighbourhood of M in T^*M has a complex structure and, considering the ball bundle with sufficiently small radius, B^*M can be considered as a strongly pseudoconvex domain with boundary S^*M. We obtain an extension of C^*-algebras

$$\mathcal{E}_{B^*M}: \quad 0 \to \mathcal{C}_0(T^*M) \to \mathcal{C}(B^*M) \to \mathcal{C}(S^*M) \to 0$$

On B^*M, there is an operator $D =: V \to V$, where V denotes the space of differential forms in $\Lambda^{0,*}$ on B^*M , satisfying a natural boundary condition on S^*M, such that the restriction of D to T^*M defines the Dolbeault operator $\bar\partial + \bar\partial^*$. Denote by H the L^2-completion of V.

Let $\bar D$ denote the maximal extension of D defined on the completion of V with respect to the norm $\|f\| = \|f + Df\|_2$. Then 0 is an isolated point in the spectrum of $\bar D$, the range of $\bar D$ is closed and its cokernel is finite-dimensional.

With respect to the decomposition of $\Lambda^{0,*}$ into even and odd forms H splits into $H = H_+ \oplus H_-$.

The Kasparov $\mathcal{C}_0(T^*M)$-$\mathcal{K}$-module (φ, F), where $F = \bar{D}/\sqrt{\bar{D}^2}$ and φ denotes the representation of $\mathcal{C}_0(T^*M)$ by multiplication operators, describes the Dolbeault element $[\bar{\partial}_{T^*M}]$. With respect to the $\mathbb{Z}/2$-grading of H and modulo compact operators F is of the form

$$\begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix}$$

where $v^*v = 1$ and $P = 1 - vv^*$ is the projection onto the Bergman space of 0-forms, thus functions u, for which $\bar{\partial}u = 0$ (i.e. holomorphic L^2-functions). Moreover, since D extends to B^*M, it satisfies the conditions required in 23 for $I = \mathcal{C}_0(T^*M)$, $A = \mathcal{C}(B^*M)$ and $B = \mathcal{C}(S^*M)$.

Therefore by the Baum-Douglas formula in 23 the image of the element $[\bar{\partial}_{T^*M}] \in KK_0(\mathcal{C}_0(T^*M), \mathbb{C})$ in $KK_0(\mathcal{C}(S^*M), \mathbb{C})$ is represented by the extension

$$0 \to \mathcal{K} \to D \to \mathcal{C}(S^*M) \to 0 \qquad (1)$$

where D denotes the subalgebra of $\mathcal{L}(PH)$ generated by $\mathcal{K} = \mathcal{K}(PH)$ together with $P\varphi(\mathcal{C}(S^*M))P$.

Boutet de Monvel [4], [3] constructs a unitary operator G mapping $L^2(M)$ to $PL^2(T^*M)$. It has the property that G^*T_fG is a pseudodifferential operator with symbol $f|_{S^*M}$ for a Toeplitz operator of the form $T_f = PfP$, $f \in \mathcal{C}(B^*M)$.

Therefore G conjugates the extension (1) of Toeplitz operators into the extension

$$\mathcal{E}_\Psi : \quad 0 \to \mathcal{K} \to \Psi \to \mathcal{C}(S^*M) \to 0$$

where Ψ denotes the C^*-completion of the algebra of pseudodifferential operators of order ≤ 0 and $\mathcal{K}$ the completion of the algebra of operators of order < 0.

In conclusion we get the theorem that

$$KK(\mathcal{E}_{B^*M})[\bar{\partial}_{T^*M}] = KK(\mathcal{E}_\Psi)$$

where $[\bar{\partial}_{T^*M}]$ is the K-homology element in $KK(\mathcal{C}_0(T^*M), \mathbb{C})$ defined by (φ, F) and where $\mathcal{E}_\Psi$, $\mathcal{E}_{B^*M}$ are the two natural extensions of $\mathcal{C}(S^*M)$.

The proof of the theorem that we outlined above works verbatim in the setting of locally convex algebras. The Baum-Douglas-Taylor theorem then reads as

$$kk(\mathcal{E}_\Psi) = kk(\mathcal{E}_{B^*M}) \cdot [\bar{\partial}_{T^*M}]$$

where $kk(\mathcal{E}_{B^*M})$, $kk(\mathcal{E}_\Psi)$, $[\bar{\partial}_{T^*M}]$ are the elements in kk_* determined by the corresponding extensions of locally convex algebras (using algebras of $\mathcal{C}^\infty$-functions and replacing the ideal $\mathcal{K}$ by a Schatten ideal).

We briefly sketch the connection of this theorem to the index theorems of Kasparov and of Atiyah-Singer. Kasparov's theorem determines the K-homology class $[P] \in KK(\mathcal{C}M, \mathbb{C})$ determined by an elliptic operator P by the formula

$$[P] = [[\sigma(P)]] \cdot [\bar{\partial}_{T^*M}]$$

Here $[[\sigma(P)]] = [[\Sigma(P)]] \cdot KK(\mathcal{E}_{B^*M})$ and $[[\Sigma(P)]] \in KK(\mathcal{C}M, \mathcal{C}(S^*M))$ is a naturally defined bivariant class associated with the symbol of P. Kasparov's formula is (in the non-equivariant case) a consequence of the Baum-Douglas-Taylor theorem, since - basically by definition - we have $[P] = [[\Sigma(P)]] \cdot KK(\mathcal{E}_\Psi)$.

The Atiyah-Singer theorem determines the index of P as an element of $KK(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$ by

$$\operatorname{ind} P = \operatorname{ind}_t[\sigma(P)]$$

where ind_t is the "topological index" map. This formula is a consequence of Kasparov's formula since, by definition, $\operatorname{ind} P = [1] \cdot [P]$, $[\sigma(P)] = [1] \cdot [[\sigma(P)]]$, and since one can check that $\operatorname{ind}_t(x) = x \cdot [\bar{\partial}_{T^*M}]$ for each x in $KK(\mathbb{C}, \mathcal{C}_0(T^*M))$.

Finally, we note that, by construction, the Baum-Douglas-Taylor theorem of course also gives a formula for the index of Toeplitz operators on strictly pseudoconvex domains. This formula is also discussed in [12].

References

1. Paul Baum and Ronald G. Douglas. Relative K homology and C^* algebras. *K-Theory*, 5(1):1–46, 1991.
2. Paul Baum, Ronald G. Douglas, and Michael E. Taylor. Cycles and relative cycles in analytic K-homology. *J. Differential Geom.*, 30(3):761–804, 1989.
3. L. Boutet de Monvel and V. Guillemin. *The spectral theory of Toeplitz operators*, volume 99 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1981.
4. Louis Boutet de Monvel. On the index of Toeplitz operators of several complex variables. *Invent. Math.*, 50(3):249–272, 1978/79.
5. J. Cuntz and G. Skandalis. Mapping cones and exact sequences in KK-theory. *J. Operator Theory*, 15(1):163–180, 1986.
6. Joachim Cuntz. Bivariante K-Theorie für lokalkonvexe Algebren und der Chern-Connes-Charakter. *Doc. Math.*, 2:139–182 (electronic), 1997.
7. Joachim Cuntz. A general construction of bivariant K-theories on the category of C^*-algebras. In *Operator algebras and operator theory (Shanghai, 1997)*, volume 228 of *Contemp. Math.*, pages 31–43. Amer. Math. Soc., Providence, RI, 1998.
8. Joachim Cuntz. Bivariant K-theory and the Weyl algebra. *K-Theory*, to appear.
9. Joachim Cuntz and Andreas Thom. Algebraic K-theory and locally convex algebras. *preprint*, 2005.
10. Nigel Higson. On the relative K-homology theory of Baum and Douglas. *preprint*, 1988.
11. Nigel Higson. C^*-algebra extension theory and duality. *J. Funct. Anal.*, 129(2):349–363, 1995.
12. Nigel Higson and John Roe. *Analytic K-homology*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000. Oxford Science Publications.

13. G. G. Kasparov. Relative K-homology and K-homology of an ideal. *K-Theory*, 5(1):47–49, 1991.
14. François Trèves. *Topological vector spaces, distributions and kernels*. Academic Press, New York, 1967.

On Rørdam's Classification of Certain C^*-Algebras with One Non-Trivial Ideal

Søren Eilers and Gunnar Restorff

Department of Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark, eilers@math.ku.dk, restorff@math.ku.dk

Summary. We extend Rørdam's stable classification result for purely infinite C^*-algebras with exactly one non-trivial ideal to allow for the lifting of an isomorphism on the level of the invariants to a $*$-isomorphism, and to allow for unital isomorphism when the isomorphisms of the invariant respect the relevant classes of units.

1 Introduction

Rørdam in [14] establishes that the six term exact sequence

$$\begin{array}{ccccc} K_0(\mathfrak{A}) & \xrightarrow{\iota_*} & K_0(\mathfrak{E}) & \xrightarrow{\pi_*} & K_0(\mathfrak{E}/\mathfrak{A}) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\partial} \\ K_1(\mathfrak{E}/\mathfrak{A}) & \xleftarrow[\pi_*]{} & K_1(\mathfrak{E}) & \xleftarrow[\iota_*]{} & K_1(\mathfrak{A}) \end{array} \tag{1}$$

is a complete invariant for stable isomorphism of C^*-algebras $\mathfrak{E}$ with precisely one non-trivial ideal $\mathfrak{A}$, provided that the ideal $\mathfrak{A}$ and the quotient $\mathfrak{E}/\mathfrak{A}$ are both in the class of purely infinite simple C^*-algebras classified by Kirchberg and Phillips ([9]).

Most classification results of C^*-algebras by K-theoretical invariants are established in such a way that one with little or no extra effort can prove that any isomorphism between a pair of invariants may be lifted to a $*$-isomorphism. It is often also easy to pass between results yielding stable isomorphism for general C^*-algebras and isomorphism of unital C^*-algebras in a certain class, by adding or leaving out the class of the unit in the invariant.

Rørdam's classification result forms a notable exception to these two rules. Indeed, there is no obvious way to extract from Rørdam's proof a way to establish these kinds of sligthly improved classification results. It is the purpose of this note to show that by invoking more recent results by Bonkat and Kirchberg, one may prove such results in the class considered by Rørdam.

Using the language promoted by Elliott ([6]) this proves that the *classification functor* used by Rørdam is indeed a *strong classification functor*.

We are first going to prove, by straightforward observations on central results in Bonkat's thesis, that every isomorphism among invariants of the type (1) – i.e., a 6-tuple of coherent group isomorphisms – lifts to a $*$-isomorphism. With this in hand we can then prove a unital classification result by appealing to a useful principle which we shall develop in a rather general context.

An update on the status of the work of Bonkat may be in order. Bonkat sets out to reprove the classification result of Rørdam using Kirchberg's results. However, the class classified by Bonkat is, a priori, smaller than the class classified by Rørdam, and to prove that they coincide in this case, Bonkat is forced to appeal to Rørdam's result. Fortunately, more recent results by Kirchberg or Toms and Winter ([16]) show in a direct way that the classes coincide, rendering Bonkat's proof truly independent of [14]. More details are given after Lemma 4.

2 Bonkat's Method

We shall concentrate on C^*-algebras $\mathfrak{E}$ having exactly one non-trivial ideal $\mathfrak{A}$, noting that this is the case exactly when the extension

$$0 \longrightarrow \mathfrak{A} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \mathfrak{E}/\mathfrak{A} \longrightarrow 0 \tag{2}$$

is essential and the C^*-algebras $\mathfrak{A}$ and $\mathfrak{E}/\mathfrak{A}$ are simple. The primitive ideal spectrum of such C^*-algebras we denote by X_1; it has two points of which the closure of one is the whole space while the other point is closed.

Kirchberg proves in [7, Corollary N] (cf. [8, Folgerung 4.3]) a result which in the case of C^*-algebras with one non-trivial ideal specializes to the following:

Theorem 1. *Let $\mathfrak{E}$ and $\mathfrak{E}'$ be strongly purely infinite, separable, stable and nuclear C^*-algebras, each with exactly one non-trivial ideal. If $z \in KK(\mathsf{X}_1; \mathfrak{E}, \mathfrak{E}')$ is a $KK(\mathsf{X}_1; -, -)$-equivalence then there exists a $*$-isomorphism*

$$\phi : \mathfrak{E} \longrightarrow \mathfrak{E}'$$

with $[\phi] = z$.

Analogously to the characterization of the bifunctor KK by universal means (cf. [1, Corollary 22.3.1]) we may describe $KK(\mathsf{X}_1; -, -)$ by the universal property that any stable, homotopy invariant and split exact functor from the category of extensions of separable C^*-algebras into an additive category factorises uniquely through $KK(\mathsf{X}_1; -, -)$. Strong pure infiniteness is considered in [11], and it is shown that a separable, stable and nuclear C^*-algebra $\mathfrak{E}$ is strongly purely infinite if and only if $\mathfrak{E}$ absorbs $\mathcal{O}_\infty$, i.e. if and only if $\mathfrak{E} \cong \mathfrak{E} \otimes \mathcal{O}_\infty$.

This extremely general and powerful result should be considered as an *isomorphism theorem* allowing one to conclude from the existence of a very weak kind of isomorphism, at the level of ideal-preserving KK-theory, the existence of a genuine $*$-isomorphism at the level of C^*-algebras.

This result could be turned into a *bona fide* classification result for such algebras with one non-trivial ideal by a suitable universal coefficient theorem allowing one to lift an isomorphism at the level of K-theory to one at the level of ideal-preserving KK. And by a very generally applicable trick originating with Rosenberg and Schochet, cf. Lemma 3 below, all one seems to need is a surjective group homomorphism

$$KK(\mathsf{X}_1; \mathfrak{E}, \mathfrak{E}') \longrightarrow \mathrm{Hom}(\mathfrak{k}(\mathfrak{E}), \mathfrak{k}(\mathfrak{E}')).$$

where $\mathfrak{k}$ is an appropriately chosen variant of K-theory. The main challenge for carrying out such a program thus becomes to identify a feasible flavour of K-theory to use as $\mathfrak{k}(-)$, and to establish the existence of such an epimorphism. However, we are aware of no approach to doing so which does not also involve identifying the kernel of this map.

Indeed, this is exactly what Bonkat manages to do in his thesis work [2] in the case when Rørdam's classification result indicates that the correct flavor of K-theory is the class of six term exact sequences considered in [14], thus providing an alternative proof for many of the results there.

A UCT for Kirchberg's $KK(\mathsf{X}_1; -, -)$ is established in [2] as follows. Bonkat works in the category of 6-periodic complexes

$$\mathbf{G} \qquad \begin{array}{ccccc} G_0 & \xrightarrow{\phi_0} & G_1 & \xrightarrow{\phi_1} & G_2 \\ {\scriptstyle \phi_5}\uparrow & & & & \downarrow{\scriptstyle \phi_2} \\ G_5 & \xleftarrow[\phi_4]{} & G_4 & \xleftarrow[\phi_3]{} & G_3 \end{array}$$

of abelian groups and group homomorphisms, which he establishes is additive. For two such complexes $\mathbf{G}$ and $\mathbf{G}'$ the natural notion of homomorphisms is the abelian group of coherent 6-tuples of group homomorphisms:

$$\mathrm{Hom}_{\hexagon}(\mathbf{G}, \mathbf{G}') = \{(\xi_i)_{i=0}^5 \mid \xi_i : G_i \longrightarrow G'_i, \phi'_i \xi_i = \xi_{i+1}\phi_i\}$$

Note that any C^*-algebra $\mathfrak{E}$ with precisely one non-trivial ideal $\mathfrak{A}$ gives rise to a 6-periodic complex

$$\begin{array}{ccccc} K_0(\mathfrak{A}) & \xrightarrow{\iota_*} & K_0(\mathfrak{E}) & \xrightarrow{\pi_*} & K_0(\mathfrak{E}/\mathfrak{A}) \\ {\scriptstyle \partial}\uparrow & & & & \downarrow{\scriptstyle \partial} \\ K_1(\mathfrak{E}/\mathfrak{A}) & \xleftarrow[\pi_*]{} & K_1(\mathfrak{E}) & \xleftarrow[\iota_*]{} & K_1(\mathfrak{A}) \end{array}$$

which we may denote $K_{\hexagon}(\mathfrak{E})$ without risk of confusion, and that any $*$-isomorphism $\phi : \mathfrak{E} \longrightarrow \mathfrak{E}'$ (as well as any other $*$-homomorphism sending $\mathfrak{A}$ into $\mathfrak{A}'$)

induces an element $\phi_* \in \mathrm{Hom}_{\hexagon}(K_{\hexagon}(\mathfrak{E}), K_{\hexagon}(\mathfrak{E}'))$. We let $K_{\hexagon+1}(\mathfrak{E})$ denote the complex obtained by shifting the groups by three indices.

Bonkat identifies the projective objects in this category as those complexes which are exact and have projective, i.e. free, groups at each entry and proves that there are enough projectives so that the $\mathrm{Hom}_{\hexagon}$-functor of coherent 6-tuples defines left derived functors $\mathrm{Ext}^n_{\hexagon}$. It is then proved ([2, Korollar 7.2.14]) that all *exact* such periodic complexes have a projective resolution of length at most one, and by giving in [2, Abschnitt 7.4] a geometric resolution – i.e. a realization at the level of C^*-algebras – of this, Bonkat arrives at the following universal coefficient theorem

Theorem 2. *[2, Cf. Satz 7.5.3] Let $\mathfrak{E}$ and $\mathfrak{E}'$ be separable C^*-algebras each with exactly one non-trivial ideal $\mathfrak{A}$ and $\mathfrak{A}'$, respectively. Assume further that $\mathfrak{A}, \mathfrak{A}', \mathfrak{E}/\mathfrak{A}$ and $\mathfrak{E}'/\mathfrak{A}'$ lie in the UCT class $\mathcal{N}$. There is a short exact sequence*

$$\mathrm{Ext}^1_{\hexagon}(K_{\hexagon}(\mathfrak{E}), K_{\hexagon+1}(\mathfrak{E}')) \longrightarrow KK(\mathsf{X}_1; \mathfrak{E}, \mathfrak{E}') \xrightarrow{\Gamma} \mathrm{Hom}_{\hexagon}(K_{\hexagon}(\mathfrak{E}), K_{\hexagon}(\mathfrak{E}'))$$

Along the way Bonkat works in a different picture of $KK(\mathsf{X}_1; \mathfrak{E}, \mathfrak{E}')$; the differences are explained in [2, Abschnitt 5.6]. By naturality of the UCT one proves as in [15, Proposition 7.3]:

Lemma 3. *[2, Proposition 7.7.2] Let $\mathfrak{E}$ and $\mathfrak{E}'$ be as in Theorem 2. The element $z \in KK(\mathsf{X}_1; \mathfrak{E}, \mathfrak{E}')$ is an equivalence precisely when*

$$\Gamma(z) \in \mathrm{Hom}_{\hexagon}(K_{\hexagon}(\mathfrak{E}), K_{\hexagon}(\mathfrak{E}'))$$

is a 6-tuple of group isomorphisms.

Following Rørdam we say that a C^*-algebra is a *Kirchberg algebra* if it is purely infinite, simple, nuclear and separable. We need to use the following:

Lemma 4. *Let $\mathfrak{E}$ be an essential extension of two stable Kirchberg algebras from the UCT class $\mathcal{N}$. Then $\mathfrak{E}$ is strongly purely infinite.*

In Bonkat's thesis ([2, Satz 7.8.8]) this is established using Rørdam's classification, but applying more recent results by Kirchberg or (using the fact that strong purely infiniteness coincides with $\mathcal{O}_\infty$-stability in this case) by Toms and Winter [16, Theorem 4.3] this may be proved directly.

Theorem 5. *Let $\mathfrak{E}$ and $\mathfrak{E}'$ be C^*-algebras each with exactly one non-trivial ideal $\mathfrak{A}$ and $\mathfrak{A}'$, with the property that $\mathfrak{A}, \mathfrak{A}', \mathfrak{E}/\mathfrak{A}, \mathfrak{E}'/\mathfrak{A}'$ are all Kirchberg algebras in the UCT class $\mathcal{N}$. Any invertible element of $\mathrm{Hom}_{\hexagon}(K_{\hexagon}(\mathfrak{E}), K_{\hexagon}(\mathfrak{E}'))$ can be realized by a $*$-isomorphism $\phi : \mathfrak{E} \otimes \mathbb{K} \longrightarrow \mathfrak{E}' \otimes \mathbb{K}$.*

Proof. We may assume that $\mathfrak{E}$ and $\mathfrak{E}'$ are themselves stable. By Lemma 3 and Theorem 2 there exists an equivalence $\gamma \in KK(\mathsf{X}_1; -, -)$ realizing this 6-tuple of morpisms. Thus by Theorem 1 and Lemma 4 the map is realized by a $*$-isomorphism $\phi : \mathfrak{E} \longrightarrow \mathfrak{E}'$.

Corollary 6. *Let $\mathfrak{E}$ be a C^*-algebra with exactly one non-trivial ideal $\mathfrak{A}$, with the property that $\mathfrak{A}$ and $\mathfrak{E}/\mathfrak{A}$ are both stable Kirchberg algebras in the UCT class $\mathcal{N}$. The map*

$$\mathrm{Aut}(\mathfrak{E}) \longrightarrow \mathrm{Aut}_{\hexagon}(K_{\hexagon}(\mathfrak{E}))$$

is surjective.

It would be interesting to investigate when two such realizing $*$-isomorphisms ϕ, ϕ' were approximately unitarily equivalent. It is necessary that ϕ and ϕ' induce the same map on $K_*(\mathfrak{E}; \mathsf{Z}/n), K_*(\mathfrak{A}; \mathsf{Z}/n)$, and $K_*(\mathfrak{B}; \mathsf{Z}/n)$ for any $n \in \{2, 3 \dots\}$, and it is tempting to conjecture that this condition is also sufficient. It is, however, not even clear that any automorphism on a six term exact sequence of total K-theory lifts to a $*$-automorphism.

3 Unital Classification

Using the main theorem of preceding section, Theorem 5, we will extend Rørdam's stable classification to allow for unital isomorphism when the isomorphisms of the invariant respect the relevant classes of units. This will be done by appealing to a useful principle which we shall develop in a rather general context. First we need some facts about properly infinite projections.

In [5] Cuntz considers C^*-algebras $\mathfrak{A}$ that contain a set $\mathscr{P}$ of projections satisfying the following conditions:

(Π_1) If $p, q \in \mathscr{P}$ and $p \perp q$, then $p + q \in \mathscr{P}$.
(Π_2) If $p \in \mathscr{P}$ and p' is a projection in $\mathfrak{A}$ such that $p \sim p'$, then $p' \in \mathscr{P}$.
(Π_3) For all $p, q \in \mathscr{P}$, there is $p' \in \mathscr{P}$ such that $p \sim p'$, $p' < q$ and $q - p' \in \mathscr{P}$.
(Π_4) If q is a projection in $\mathfrak{A}$, which majorizes an element of $\mathscr{P}$, then $q \in \mathscr{P}$.

If p is a projection in a C^*-algebra, then we let $[p]$ denote the Murray–von Neumann equivalence class of this projection. Cuntz shows in [5, Theorem 1.4] the following theorem:

Theorem 7. *Let $\mathfrak{A}$ be a C^*-algebra with a non-empty set $\mathscr{P} \subseteq \mathfrak{A}$ of projections satisfying (Π_1), (Π_2) and (Π_3) above. Then $G = \{[p] \mid p \in \mathscr{P}\}$ is a group with the natural addition $[p]+[q] = [p'+q']$, where $p', q' \in \mathscr{P}$ are chosen such that $p \sim p'$, $q \sim q'$ and $p' \perp q'$ by (Π_3). Moreover, if $\mathfrak{A}$ is unital and $\mathscr{P}$ also satisfies (Π_4), then $G \ni [p] \mapsto [p]_0 \in K_0(\mathfrak{A})$ defines a group isomorphism.*

Recall that a projection p in a C^*-algebra $\mathfrak{A}$ is called *full* if $\mathfrak{A}$ is the only ideal in $\mathfrak{A}$ containing p, and p is called *properly infinite* if there exist projections $p_1, p_2 \leq p$ in $\mathfrak{A}$ such that $p_1 \perp p_2$ and $p_1 \sim p_2 \sim p$. See e.g. [10] and [11] for more on infinite projections and related topics.

Lemma 8. *Let $\mathfrak{A}$ be a C^*-algebra and let $\mathscr{P}$ be the set of full, properly infinite projections in $\mathfrak{A}$. Then $\mathscr{P}$ satisfies (Π_1), (Π_2), (Π_3) and (Π_4).*

Proof. (Π_1): Suppose there are given projections $p, q \in \mathscr{P}$ with $p \perp q$. Then there exist projections p_1, p_2, q_1, q_2 in $\mathfrak{A}$ such that

$$p_1, p_2 \leq p, \quad q_1, q_2 \leq q, \quad p_1 \perp p_2, \quad q_1 \perp q_2, \quad p_1 \sim p_2 \sim p, \quad q_1 \sim q_2 \sim q.$$

Put $r_1 = p_1 + q_1$, $r_2 = p_2 + q_2$ and $r = p + q$. It is easy to check that these are projections satisfying $r_1, r_2 \leq r$, $r_1 \perp r_2$ and $r_1 \sim r_2 \sim r$; i.e. r is properly infinite. Clearly r is full, so $r \in \mathscr{P}$.

(Π_2): Let there be given projections $p \in \mathscr{P}$ and $p' \in \mathfrak{A}$ such that $p \sim p'$. Then there exist orthogonal projections $p_1, p_2 \leq p$, such that $p_1 \sim p_2 \sim p$, and there exists a partial isometry $v \in \mathfrak{A}$ such that $p = vv^*$ and $p' = v^*v$. Define $p_1' = v^*p_1v$ and $p_2' = v^*p_2v$. Then one easily shows, that p_1' and p_2' are orthogonal projections such that $p_1', p_2' \leq p'$ and $p_1' \sim p_2' \sim p'$. From $p = p^2 = vv^*vv^* = vp'v^*$ it is clear that p' is full. Hence $p' \in \mathscr{P}$

(Π_4): Let q be a projection in $\mathfrak{A}$ such that $p \leq q$ for a $p \in \mathscr{P}$. Then $p \precsim q$, and hence q is properly infinite by [10, Lemma 3.8] (see Section 2 in the same paper for more on Cuntz comparison $\precsim$). From $p \leq q$ we immidiately get that $pq = p$, so q is clearly full. Thus we have shown that $q \in \mathscr{P}$.

(Π_3): Let $p, q \in \mathscr{P}$ be given projections. Then the ideal $\overline{\mathfrak{A}q\mathfrak{A}}$ generated by q is $\mathfrak{A}$ (q is full). According to [10, Proposition 3.5] we have $p \precsim q$, i.e. there exists a projection $p' \leq q$ such that $p \sim p'$. So there exist orthogonal projections $p_1', p_2' \leq p'$ in $\mathfrak{A}$ such that $p_1' \sim p_2' \sim p'$. The projection p is in $\mathscr{P}$, which by (Π_2) implies that $p_1', p_2' \in \mathscr{P}$. From $p_1' + p_2' \leq p' \leq q$ we deduce that $p_2' \leq q - p_1' < q$. From (Π_4) we get $q - p_1' \in \mathscr{P}$, because $p_2' \in \mathscr{P}$.

Analogous to Brown's result ([3, Corollary 2.7]) one easily proves the following theorem:

Theorem 9. *Let p be a full projection in a separable C^*-algebra $\mathfrak{A}$. Then the embedding $\iota\colon p\mathfrak{A}p \to \mathfrak{A}$ induces an isomorphism $K_0(\iota)\colon K_0(p\mathfrak{A}p) \to K_0(\mathfrak{A})$.*

Proposition 10. *Let p and q be full, properly infinite projections in a separable C^*-algebra $\mathfrak{A}$. Then $[p]_0 = [q]_0$ if and only if p is Murray–von Neumann equivalent to q.*

Proof. Let p and q be full, properly infinite projections in a separable C^*-algebra $\mathfrak{A}$. Assume that $[p]_0 = [q]_0$. We want to show, that $p \sim q$. By (Π_3) we can w.l.o.g. assume that $p \perp q$. Put $r = p + q$.

The hereditary corner algebra $r\mathfrak{A}r$ of $\mathfrak{A}$ is unital. The set $\mathscr{P}$ of full, properly infinite projections in $r\mathfrak{A}r$ contains p and q. By Theorem 9, $[p]_0 = [q]_0$ in $K_0(r\mathfrak{A}r)$. By Cuntz' result is $p \sim q$ (in $r\mathfrak{A}r$).

The claims in this proposition are stated several places in the literature for unital C^*-algebras without the separability condition, but the proofs do not readily generalize to the non-unital case. It is likely that one can get by without the separability condition – it may even be a known result – but we will not need this here.

We can use Cuntz' argument in the proof of [13, Theorem 6.5] to prove the following meta-theorem:

Theorem 11. *Let $\mathscr{C}$ be a subcategory of the category of C^*-algebras, and let $\mathsf{F}\colon \mathscr{C} \to \mathscr{D}$ be a covariant functor defined on this subcategory. Assume that*

(i) For every C^*-algebra $\mathfrak{A}$ in $\mathscr{C}$, $\mathfrak{A}\otimes\mathbb{K}$ belongs to $\mathscr{C}$, and the $*$-homomorphism $\mathfrak{A} \ni a \mapsto a \otimes e \in \mathfrak{A}\otimes\mathbb{K}$ induces an isomorphism from $\mathsf{F}(\mathfrak{A})$ onto $\mathsf{F}(\mathfrak{A}\otimes\mathbb{K})$, where e is a minimal projection in $\mathbb{K}$.

(ii) For all stable C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathscr{C}$, every isomorphism from $\mathsf{F}(\mathfrak{A})$ to $\mathsf{F}(\mathfrak{B})$ is induced by a $*$-isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

(iii) There exists a covariant functor G from $\mathscr{D}$ into the category of abelian groups such that $\mathsf{G} \circ \mathsf{F} = K_0$

Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital, properly infinite, separable C^*-algebras from $\mathscr{C}$. If there exists an isomorphism ρ from $\mathsf{F}(\mathfrak{A})$ onto $\mathsf{F}(\mathfrak{B})$, such that $\mathsf{G}(\rho)$ maps $[\mathbb{1}_{\mathfrak{A}}]_0$ onto $[\mathbb{1}_{\mathfrak{B}}]_0$, then the C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are $*$-isomorphic. (If $\mathfrak{A}\otimes\mathbb{K}$ and $\mathfrak{B}\otimes\mathbb{K}$ have the cancellation property, we may omit the assumption of the algebras being properly infinite.)

Proof. Let $\rho\colon \mathsf{F}(\mathfrak{A}) \to \mathsf{F}(\mathfrak{B})$ be an isomorphism such that $\alpha = \mathsf{G}(\rho)$ maps $[\mathbb{1}_{\mathfrak{A}}]_0$ onto $[\mathbb{1}_{\mathfrak{B}}]_0$, i.e. $\alpha([\mathbb{1}_{\mathfrak{A}}]_0) = [\mathbb{1}_{\mathfrak{B}}]_0$. Let e denote a minimal projection in $\mathbb{K}$. The homomorphisms $\mathfrak{A} \ni a \mapsto a \otimes e \in \mathfrak{A}\otimes\mathbb{K}$ and $\mathfrak{B} \ni b \mapsto b \otimes e \in \mathfrak{B}\otimes\mathbb{K}$ induce isomorphisms from $\mathsf{F}(\mathfrak{A})$ to $\mathsf{F}(\mathfrak{A}\otimes\mathbb{K})$ and from $\mathsf{F}(\mathfrak{B})$ to $\mathsf{F}(\mathfrak{B}\otimes\mathbb{K})$, resp. Therefore we get an induced isomorphism $\tilde{\rho}$ from $\mathsf{F}(\mathfrak{A}\otimes\mathbb{K})$ to $\mathsf{F}(\mathfrak{B}\otimes\mathbb{K})$, with $\tilde{\alpha} = \mathsf{G}(\tilde{\rho})$ being an isomorphism from $K_0(\mathfrak{A}\otimes\mathbb{K})$ to $K_0(\mathfrak{B}\otimes\mathbb{K})$ such that $\tilde{\alpha}([\mathbb{1}_{\mathfrak{A}} \otimes e]_0) = [\mathbb{1}_{\mathfrak{B}} \otimes e]_0$.

By assumption, $\tilde{\rho}$ (and therefore also $\tilde{\alpha}$) is induced by a $*$-isomorphism $\phi\colon \mathfrak{A}\otimes\mathbb{K} \to \mathfrak{B}\otimes\mathbb{K}$. So

$$[\phi(\mathbb{1}_{\mathfrak{A}} \otimes e)]_0 = K_0(\phi)([\mathbb{1}_{\mathfrak{A}} \otimes e]_0) = \tilde{\alpha}([\mathbb{1}_{\mathfrak{A}} \otimes e]_0) = [\mathbb{1}_{\mathfrak{B}} \otimes e]_0.$$

The projections $\phi(\mathbb{1}_{\mathfrak{A}} \otimes e)$ and $\mathbb{1}_{\mathfrak{B}} \otimes e$ are full and properly infinite – we show this only for $\mathbb{1}_{\mathfrak{B}} \otimes e$ (ϕ is a $*$-isomorphism). It is clear that $\mathbb{1}_{\mathfrak{B}} \otimes e$ is a full projection. The projection $\mathbb{1}_{\mathfrak{B}}$ is properly infinite, so there exist partial isometries u_1 and u_2 such that $u_1u_1^* = u_2u_2^* = \mathbb{1}_{\mathfrak{B}}$ and $u_1^*u_1 \perp u_2^*u_2$; from this we see that $(u_1 \otimes e)(u_1 \otimes e)^* = \mathbb{1}_{\mathfrak{B}} \otimes e = (u_2 \otimes e)(u_2 \otimes e)^*$ and $(u_1 \otimes e)^*(u_1 \otimes e)(u_2 \otimes e)^*(u_2 \otimes e) = u_1^*u_1u_2^*u_2 \otimes e = 0$. We have thus shown that the projection is properly infinite. By Proposition 10, therefore $\phi(\mathbb{1}_{\mathfrak{A}} \otimes e)$ is Murray–von Neumann equivalent to $\mathbb{1}_{\mathfrak{B}} \otimes e$. So there exists a partial isometry v such that

$$vv^* = \mathbb{1}_{\mathfrak{B}} \otimes e \quad \text{and} \quad v^*v = \phi(\mathbb{1}_{\mathfrak{A}} \otimes e).$$

Then $x \otimes e \mapsto v\phi(x \otimes e)v^*$ is a $*$-isomorphism from $\mathfrak{A} \otimes \mathbb{C}e$ onto $\mathfrak{B} \otimes \mathbb{C}e$. Because it is

- *well-defined:* For all $x \in \mathfrak{A}$ is

$$v\phi(x \otimes e)v^* = (\mathbb{1}_{\mathfrak{B}} \otimes e)v\phi(x \otimes e)v^*(\mathbb{1}_{\mathfrak{B}} \otimes e) \in \mathfrak{B} \otimes \mathbb{C}e.$$

- *a homomorphism:* The map $x \otimes e \mapsto v\phi(x \otimes e)v^*$ is clearly linear and $*$-preserving. For $x, y \in \mathfrak{A}$ is

  $$v\phi(xy \otimes e)v^* = v\phi(x \otimes e)\phi(\mathbb{1}_{\mathfrak{A}} \otimes e)\phi(y \otimes e)v^* = v\phi(x \otimes e)v^* v\phi(y \otimes e)v^*.$$

- *surjective:* Let $y \in \mathfrak{B}$ be given. Then there exists $x \in \mathfrak{A} \otimes \mathbb{K}$ such that $\phi(x) = v^*(y \otimes e)v$. So

  $$v\phi((\mathbb{1}_A \otimes e)x(\mathbb{1}_{\mathfrak{A}} \otimes e))v^* = v\phi(x)v^* = vv^*(y \otimes e)vv^* = y \otimes e.$$

  Because $(\mathbb{1}_A \otimes e)x(\mathbb{1}_{\mathfrak{A}} \otimes e) \in \mathfrak{A} \otimes \mathbb{C}e$ the homomorphism is surjective.
- *injective:* Let $x, y \in \mathfrak{A}$. If $v\phi(x \otimes e)v^* = v\phi(y \otimes e)v^*$, then

  $$\begin{aligned}\phi(x \otimes e) &= \phi(\mathbb{1}_{\mathfrak{A}} \otimes e)\phi(x \otimes e)\phi(\mathbb{1}_{\mathfrak{A}} \otimes e) = v^* v\phi(x \otimes e)v^* v \\ &= v^* v\phi(y \otimes e)v^* v = \phi(\mathbb{1}_{\mathfrak{A}} \otimes e)\phi(y \otimes e)\phi(\mathbb{1}_{\mathfrak{A}} \otimes e) = \phi(y \otimes e)\end{aligned}$$

  and, consequently, $x = y$.

Corollary 12. *Let $\mathfrak{A}, \mathfrak{A}', \mathfrak{B}$ and $\mathfrak{B}'$ be Kirchberg algebras from the UCT class $\mathcal{N}$, and assume that $\mathfrak{E}$ and $\mathfrak{E}'$ are unital, essential extensions:*

$$0 \longrightarrow \mathfrak{A} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \mathfrak{B} \longrightarrow 0$$

$$0 \longrightarrow \mathfrak{A}' \xrightarrow{\iota'} \mathfrak{E}' \xrightarrow{\pi'} \mathfrak{B}' \longrightarrow 0.$$

Then $\mathfrak{E} \cong \mathfrak{E}'$ if and only if there exists an isomorphism between the six term exact sequences from K-theory mapping $[\mathbb{1}_{\mathfrak{E}}]_0$ onto $[\mathbb{1}_{\mathfrak{E}'}]_0$.

Proof. By [14, Proposition 4.1] $\mathfrak{E}$ and $\mathfrak{E}'$ are properly infinite. This Corollary follows now directly from the Theorems 5 and 11 (where the objects of the subcategory are the C^*-algebras, which are essential extensions of Kirchberg algebras from the UCT class $\mathcal{N}$, and the morphisms are the $*$-homomorphisms mapping the non-trivial essential ideal into the non-trivial essential ideal).

Let $0 \to \mathfrak{A} \to \mathfrak{E} \to \mathfrak{B} \to 0$ be an essential extension of (non-zero) Kirchberg algebras. It is well known that Kirchberg algebras are either stable or unital. This forces $\mathfrak{A}$ to be stable. Then, as pointed out in [14], there are three kinds of extensions: (i) $\mathfrak{E}$ (and hence $\mathfrak{B}$) is unital, (ii) $\mathfrak{B}$ is unital but $\mathfrak{E}$ has no unit, and (iii) $\mathfrak{B}$ (and hence $\mathfrak{E}$) has no unit. In the latter case, both $\mathfrak{E}$ and $\mathfrak{B}$ are stable. Assuming that the algebras belong to the UCT class $\mathcal{N}$, we have classified the algebras of the first type up to $*$-isomorphism, while Rørdam has classified the algebras of the third type up to $*$-isomorphism. What remains is to classify the algebras in the intermediate case, where $\mathfrak{E}$ is neither unital nor stable[1].

[1]Note added in proof: This problem has been solved by the second named author and Efren Ruiz in *On Rørdam's classification of certain C^*-algebras with one non-trivial ideal, II*, preprint, 2006. The range question for the case considered in the present paper is also addressed there.

Corollary 13. *Let A and A' be non-degenerate $\{0,1\}$-matrices in the following block form*

$$A = \begin{pmatrix} M & 0 \\ X & N \end{pmatrix}, \quad A' = \begin{pmatrix} M' & 0 \\ X' & N' \end{pmatrix},$$

where N and N' are irreducible non-permutation matrices, the maximal non-degenerate principal submatrices of M and M' are irreducible non-permutationmatrices, $X \neq 0$, and $X' \neq 0$. So the matrices A and A' satify condition (II) of Cuntz ([4]) and the Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_{A'}$ have exactly one non-trivial closed ideal.

Then $\mathcal{O}_A \cong \mathcal{O}_{A'}$ if and only if there exist isomorphisms

$$\gamma_1 \colon \ker(I - N^{\mathsf{T}}) \to \ker(I - {N'}^{\mathsf{T}}),$$

$$\alpha_0 \colon \operatorname{cok}(I - M^{\mathsf{T}}) \to \operatorname{cok}(I - {M'}^{\mathsf{T}}),$$

$$\beta_0 \colon \operatorname{cok}(I - A^{\mathsf{T}}) \to \operatorname{cok}(I - {A'}^{\mathsf{T}})$$

such that

$$\begin{array}{ccccc}
\ker(I - N^{\mathsf{T}}) & \xrightarrow{y \mapsto [X^{\mathsf{T}} y]} & \operatorname{cok}(I - M^{\mathsf{T}}) & \xrightarrow{[x] \mapsto \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]} & \operatorname{cok}(I - A^{\mathsf{T}}) \\
\cong \downarrow \gamma_1 & & \cong \downarrow \alpha_0 & & \cong \downarrow \beta_0 \\
\ker(I - {N'}^{\mathsf{T}}) & \xrightarrow{y \mapsto [{X'}^{\mathsf{T}} y]} & \operatorname{cok}(I - {M'}^{\mathsf{T}}) & \xrightarrow{[x] \mapsto \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]} & \operatorname{cok}(I - {A'}^{\mathsf{T}})
\end{array}$$

commutes and $\beta_0([1\,1 \cdots 1]^{\mathsf{T}}) = [1\,1 \cdots 1]^{\mathsf{T}}$.

Proof. This follows from the previous Corollary combined with the paper [12] – the invariant there also asks for an isomorphism between the K_0-groups of the quotients, but here the existence is automatic, and no other commutative diagrams are required (because we have only *one* non-trivial ideal).

References

1. B. Blackadar, K-Theory for Operator Algebras. *Math. Sci. Research Inst. Publ.* **5**, Springer-Verlag, New York, 1986.
2. A. Bonkat, *Bivariante K-Theorie für Kategorien projektiver Systeme von C^*-Algebren*, Ph.D. thesis, Westfälische Wilhelms-Universität, 2002, `http://wwwmath1.uni-muenster.de/sfb/about/publ/heft319.ps`.
3. L.G. Brown, *Stable isomorphism of hereditary subalgebras of C^*-algebras*, Pacific J. Math. **71** (1977), 335–348.
4. J. Cuntz, *A class of C^*-algebras and topological markov chains II: Reducible chains and the Ext-functor for C^*-algebras*, Invent. Math. **63** (1981), 25–40.
5. J. Cuntz, *K-theory for certain C^*-algebras*, Ann. of Math. **113** (1981), 181–197.
6. G. A. Elliott, *Towards a theory of classification*, in preparation.

7. E. Kirchberg, *The classification of purely infinite C^*-algebras using Kasparov's theory*, preprint, third draft, 1994.
8. E. Kirchberg, *Das nicht-kommutative Michael-Auswahlprinzip und die Klassifikation nicht-einfacher Algebren*, C^*-algebras (Münster, 1999), Springer, Berlin, 2000, pp. 92–141.
9. E. Kirchberg and N. C. Phillips, *Embedding of exact C^*-algebras in the Cuntz algebra $\mathcal{O}_2$*, J. Reine Angew. Math. **525** (2000), 17–53.
10. E. Kirchberg and M. Rørdam, *Non-simple purely infinite C^*-algebras*, Amer. J. Math. **122** (2000), 637–666.
11. E. Kirchberg and M. Rørdam, *Infinite non-simple C^*-algebras: absorbing the Cuntz algebras $\mathcal{O}_\infty$*, Adv. Math. **167** (2002), 195–264.
12. G. Restorff, *Classification of Cuntz-Krieger algebras up to stable isomorphism*, *J. Reine Angew. Math.*, in press.
13. M. Rørdam, *Classification of Cuntz-Krieger algebras*, K-Theory **9** (1995), 31–58.
14. M. Rørdam, *Classification of extensions of certain C^*-algebras by their six term exact sequences in K-theory*, Math. Ann. **308** (1997), 93–117.
15. J. Rosenberg and C. Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor*, Duke Math. J. **55** (1987), 431–474.
16. A. S. Toms and W. Winter, *Strongly self-absorbing C^*-algebras*, preprint, 2005, OA.0502211. Trans. Amer. Math. Soc. (to appear).

Perturbation of Hausdorff Moment Sequences, and an Application to the Theory of C*-Algebras of Real Rank Zero

George A. Elliott[1] and Mikael Rørdam[2]

[1] Department of Mathematics, University of Toronto, Toronto, Canada M5S 3G3, elliott@math.toronto.edu
[2] Department of Mathematics, University of Southern Denmark, 5230 Odense M, Denmark, mikael@imada.sdu.dk

Summary. We investigate the class of unital C*-algebras that admit a unital embedding into *every* unital C*-algebra of real rank zero, that has no finite-dimensional quotients. We refer to a C*-algebra in this class as an *initial object*. We show that there are many initial objects, including for example some unital, simple, infinite-dimensional AF-algebras, the Jiang-Su algebra, and the GICAR-algebra.

That the GICAR-algebra is an initial object follows from an analysis of Hausdorff moment sequences. It is shown that a dense set of Hausdorff moment sequences belong to a given dense subgroup of the real numbers, and hence that the Hausdorff moment problem can be solved (in a non-trivial way) when the moments are required to belong to an arbitrary simple dimension group (i.e., unperforated simple ordered group with the Riesz decomposition property).

1 Introduction

The following three questions concerning an arbitrary unital C*-algebra A, that is "large" in the sense that it has no finite-dimensional representation, are open.

Question 1. Does A contain a simple, unital, infinite-dimensional sub-C*-algebra?

Question 2 (The Global Glimm Halving Problem). Does A contain a full[3] sub-C*-algebra isomorphic to $C_0((0,1], M_2)$?

Question 3. Is there a unital embedding of the Jiang-Su algebra $\mathcal{Z}$ into A?

[3] A subset of a C*-algebra is called *full* if it is not contained in any proper closed two-sided ideal of the C*-algebra.

The Jiang-Su algebra (see [7]) is a simple, unital, infinite-dimensional C*-algebra, which is KK-equivalent to the complex numbers (and hence at least from a K-theoretical point of view could be an initial object as suggested in Question 3). An affirmative answer to Question 3 clearly would yield an affirmative answer to Question 1. A version of a lemma of Glimm (see [9, Proposition 4.10]) confirms Question 2 in the special case that A is simple (and not isomorphic to $\mathbb{C}$); so Question 2 is weaker than Question 1.

Question 2 was raised in [10, Section 4] because a positive answer will imply that every weakly purely infinite C*-algebra is automatically purely infinite.

The Jiang-Su algebra plays a role in the classification program for amenable C*-algebras (a role that may well become more important in the future). An affirmative answer to Question 3 will, besides also answering the two other questions, shed more light on the Jiang-Su algebra. It would for example follow that the Jiang-Su algebra is the (necessarily unique) unital, simple, separable infinite-dimensional C*-algebra with the property stipulated in Question 3 and with the property (established in [7]) that every unital endomorphism can be approximated in the pointwise-norm topology by inner automorphisms.

We provide in this paper an affirmative answer to the three questions above in the special case in which the target C*-algebra A is required to be of real rank zero (in addition to being unital and with no finite-dimensional representations).

Zhang proved in [14] that in any unital simple non-elementary C*-algebra of real rank zero and for any natural number n one can find pairwise orthogonal projections $p_0, p_1, \dots, p_n$ that sum up to 1 and satisfy $p_0 \precsim p_1 \sim p_2 \sim \cdots \sim p_n$. In other words, one can divide the unit into $n+1$ pieces where n of the pieces are of the same size, and the last piece is smaller. This result was improved in [11] where it was shown that for every natural number n one can unitally embed $M_n \oplus M_{n+1}$ into any unital C*-algebra of real rank zero, that has no non-zero representation of dimension $< n$. Thus, in the terminology of the abstract, $M_n \oplus M_{n+1}$ is an initial object for every n. We shall here extend this result and show that also the infinite tensor product $P = \bigotimes_{n=1}^{\infty} M_2 \oplus M_3$ is an initial object.

We shall give an algorithm which to an arbitrary unital AF-algebra, that has no finite-dimensional representations, assigns a unital *simple* infinite-dimensional AF-algebra that embeds unitally into A. This leads to the existence of a unital infinite-dimensional simple AF-algebra that unitally embeds into P, and hence is an initial object. The Jiang-Su algebra was shown by Jiang and Su to embed unitally into any unital simple non-elementary AF-algebra, and so is also an initial object.

In Section 4 we shall show that the Gauge Invariant CAR-algebra is an initial object. Along the way to this result we shall prove a perturbation result that may be of independent interest: the set of Hausdorff moment sequences, with the property that all terms belong to an arbitrary fixed dense subset of the real numbers, is a dense subset of the Choquet simplex of all Hausdorff moment sequences.

We shall show in Section 5 that a simple, unital, infinite-dimensional C*-algebra of real rank zero must have infinite-dimensional trace simplex if it is an initial object. This leads to the open question if one can characterise initial objects among (simple) unital infinite-dimensional C*-algebras of real rank zero (or among simple AF-algebras).

We hope that our results will find application in the future study of real rank zero C*-algebras; and we hope to have cast some light on the three fundamental questions raised above.

2 Initial Objects in Unital Real Rank Zero C^*-Algebras

Definition 4. *A unital C*-algebra A will be said in this paper to be an* initial object *if it embeds unitally into any unital C*-algebra of real rank zero which has no non-zero finite-dimensional representations. (Note that we do not require A to belong to the class of algebras with these properties.) (Also we do not require the embedding to be unique in any way.)*

It is clear that the algebra of complex numbers $\mathbb{C}$ is an initial object, even in the category of all unital C*-algebras—and that it is the unique initial object in this larger category. It will be shown in Proposition 6 below that the infinite C*-algebra tensor product $P = \bigotimes_{n=1}^{\infty} M_2 \oplus M_3$ is also an initial object in the sense of the present paper. Note that this C*-algebra in fact belongs to the category we are considering, i.e., unital C*-algebras of real rank zero with no non-zero finite-dimensional quotients. It follows that a C*-algebra is an initial object in our sense if and only if it is (isomorphic to) a unital sub-C*-algebra of P. We shall use this fact to exhibit a perhaps surprisingly large number of initial objects, including many simple AF-algebras, the Jiang-Su algebra, and the GICAR-algebra (the gauge invariant subalgebra of the CAR-algebra).

Let us begin by recalling the following standard fact.

Lemma 5. *Let A be a unital C*-algebra and let F be a unital finite-dimensional sub-C*-algebra of A. Let $g_1, \ldots, g_n$ denote the minimal (non-zero) central projections in F and let $e_1, \ldots, e_n$ be minimal (non-zero) projections in $Fg_1, \ldots, Fg_n$, respectively.*

The map consisting of multiplying by the sum $e_1 + \cdots + e_n$ is an isomorphism from the relative commutant $A \cap F'$ of F in A onto the sub-C-algebra $e_1Ae_1 \oplus e_2Ae_2 \oplus \cdots \oplus e_nAe_n$ of A. Moreover, if B is another unital C*-algebra and $\rho_j \colon B \to e_jAe_j$ are unital *-homomorphisms, then there is a unique unital *-homomorphism $\rho \colon B \to A \cap F'$ such that $\rho(b)e_j = e_j\rho(b) = \rho_j(b)$ for all $b \in B$ and all $j = 1, \ldots, n$.*

Proposition 6. *The C*-algebra $P = \bigotimes_{n=1}^{\infty} M_2 \oplus M_3$ is an initial object (in the sense of Definition 4).*

Proof. Let A be a unital C*-algebra of real rank zero with no non-zero finite-dimensional representations. We must find a unital embedding of P into A.

Set $\bigotimes_{j=1}^{n} M_2 \oplus M_3 = P_n$, so that $P_{n+1} = P_n \otimes (M_2 \oplus M_3)$. Let us construct for each n a unital embedding $\varphi_n \colon P_n \to A$ in such a way that $\varphi_{n+1}(x \otimes 1) = \varphi_n(x)$ for each $x \in P_n$. This will yield a unital embedding of P into A as desired. In order to be able construct these maps inductively we must require in addition that they be full.[4]

For each full projection e in A there is a full unital embedding $\psi \colon M_2 \oplus M_3 \to eAe$. Indeed, eAe cannot have any non-zero finite-dimensional representation since any such representation would extend to a finite-dimensional representation of A (on a larger Hilbert space). Hence by [11, Proposition 5.3], there is a unital *-homomorphism from $M_5 \oplus M_7$ into eAe. Composing this with a full unital embedding $M_2 \oplus M_3 \to M_5 \oplus M_7$ yields the desired full embedding ψ.

The preceding argument shows that there is a full unital embedding $\varphi_1 \colon P_1 = M_2 \oplus M_3 \to A$. Suppose that $n \geq 1$ and that maps $\varphi_1, \varphi_2, \dots, \varphi_n$ have been found with the desired properties.

Choose minimal projections $f_1, f_2, \dots, f_{2^n}$ in P_n, one in each minimal non-zero direct summand, and set $\varphi_n(f_j) = e_j$. Each e_j is then a full projection in A. Choose a full unital embedding $\rho_j \colon M_2 \oplus M_3 \to e_j A e_j$ for each j, and note that by Lemma 5 there exists a a unital *-homomorphism $\rho \colon M_2 \oplus M_3 \to A \cap \varphi_n(P_n)'$ such that $\rho(b)e_j = e_j\rho(b) = \rho_j(b)$ for all $b \in M_2 \oplus M_3$ and all j. There is now a unique *-homomorphism $\varphi_{n+1} \colon P_{n+1} = P_n \otimes (M_2 \oplus M_3) \to A$ with the property that $\varphi_{n+1}(a \otimes b) = \varphi_n(a)\rho(b)$ for $a \in P_n$ and $b \in M_2 \oplus M_3$. To show that φ_{n+1} is full it suffices to check that $\varphi_{n+1}(f_j \otimes b)$ is full in A for all j and for all non-zero b in $M_2 \oplus M_3$; this follows from the identity $\varphi_{n+1}(f_n \otimes b) = \varphi_n(f_j)\rho(b) = e_j\rho(b) = \rho_j(b)$ and the fact that ρ_j is full. □

Corollary 7. *Let A be a unital C*-algebra of real rank zero. The following three conditions are equivalent.*

1. *A has no non-zero finite-dimensional representations.*
2. *There is a unital embedding of $\bigotimes_{n=1}^{\infty} M_2 \oplus M_3$ into A.*
3. *There is a unital embedding of each initial object*[5] *into A.*

Proof. (i) ⇒ (iii) is true by Definition 4. (iii) ⇒ (ii) follows from Proposition 6. (ii) ⇒ (i) holds because any finite-dimensional representation of A would restrict to a finite-dimensional representation of $\bigotimes_{n=1}^{\infty} M_2 \oplus M_3$, and no such exists. □

As remarked above, a C*-algebra is an initial object if and only if it embeds unitally into the C*-algebra $P = \bigotimes_{n=1}^{\infty} M_2 \oplus M_3$. The ordered K_0-group of P can be described as follows. Consider the Cantor set $X = \prod_{n=1}^{\infty}\{0,1\}$.

[4] By a full *-homomorphism we mean a *-homomorphism that maps each non-zero element to a full element in the codomain algebra. (A full element is one not belonging to any proper closed two-sided ideal.)

[5] The list of initial objects includes some simple unital infinite-dimensional AF-algebras and the Jiang-Su algebra $\mathcal{Z}$ as shown in Section 2.

Consider the maps $\nu_0, \nu_1 \colon X \to \mathbb{N}_0 \cup \{\infty\}$ that for each $x \in X$ count the number of 0s and 1s, respectively, among the coordinates of x, and note that $\nu_0(x) + \nu_1(x) = \infty$ for every $x \in X$. For each supernatural number n denote by $\mathcal{Q}(n)$ the set of rational numbers p/q with q dividing n, and consider the subgroup $G \subseteq C(X, \mathbb{R})$ consisting of those functions g for which $g(x) \in \mathcal{Q}(2^{\nu_0(x)} 3^{\nu_1(x)})$ for every $x \in X$. Equip G with the pointwise order, i.e., $g \geq 0$ if $g(x) \geq 0$ for all $x \in X$. Then $(K_0(P), K_0(P)^+, [1])$ is isomorphic to $(G, G^+, 1)$. Note in particular that G is a dense subgroup of $C(X, \mathbb{R})$.

3 Simple Initial Objects

We shall show in this section that the class of initial objects, in the sense of the previous section, includes several simple unital (infinite-dimensional) AF-algebras.

Lemma 8. *The following two conditions are equivalent for any dimension group G.*

1. *For each order unit x in G there exists an order unit y in G such that $2y \leq x$.*
2. *For each finite set of order units $x_1, \dots, x_k$ in G and for each set of natural numbers $n_1, \dots, n_k$ there is an order unit y in G such that $n_j y \leq x_j$ for $j = 1, 2, \dots, k$.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from the well-known fact (which is also easy to prove—using the Effros-Handelman-Shen theorem) that if $x_1, x_2, \dots, x_k$ are order units in a dimension group G, then there is an order unit y_0 in G such that $y_0 \leq x_j$ for all j. The implication (ii) $\Rightarrow$ (i) is immediate. ☐

A dimension group will be said to have the property (D) if it satisfies the two equivalent conditions of Lemma 8.

Lemma 9. *Let A be a unital AF-algebra. The ordered group $K_0(A)$ has the property (D) if and only if A has no non-zero finite-dimensional representations.*

Proof. Suppose that A has no non-zero finite-dimensional representation, and let x be an order unit in $K_0(A)$. Then $x = [e]$ for some full projection e in $M_n(A)$ for some n. Since any finite-dimensional representation of $eM_n(A)e$ would induce a finite-dimensional representation of A (on a different Hilbert space), $eM_n(A)e$ has no non-zero finite-dimensional representation. By [11, Proposition 5.3] there is a unital *-homomorphism from $M_2 \oplus M_3$ into $eM_n(A)e$. (Cf. proof of Proposition 6 above.) Let $f = (f_1, f_2)$ be a projection in $M_2 \oplus M_3$, with f_1 and f_2 one-dimensional, and denote by $\tilde{f} \in eM_n(A)e$ the image of f under the unital *-homomorphism $M_2 \oplus M_3 \to eM_n(A)e$. Then $\tilde{f}$ is full in $eM_n(A)e$ (because f is full in $M_2 \oplus M_3$), and $2[\tilde{f}] \le [e]$, as desired.

Suppose conversely that $K_0(A)$ has the property (D). Condition 2.1 (ii) with $k = 1$ implies immediately that every non-zero representation of A is infinite-dimensional. □

We present below a more direct alternative proof (purely in terms of ordered group theory) of the first implication of the lemma above. Consider a decomposition of $K_0(A)$ as the ordered group inductive limit of a sequence of ordered groups $G_1 \to G_2 \to \cdots$ with each G_i isomorphic to a finite ordered group direct sum of copies of $\mathbb{Z}$, and let x be an order unit in $K_0(A)$. Modifying the inductive limit decomposition of $K_0(A)$, we may suppose that x is the image of an order unit x_1 in G_1, and that the image x_n of x_1 in G_n is an order unit for G_n for each $n \ge 2$. Let us show that for some n the condition 2.1 (i) holds for x_n in G_n—or else, if not, then G has a non-zero quotient ordered group isomorphic to $\mathbb{Z}$. If not, then for every n there exists at least one coordinate of x_n in G_n equal to one, and the inductive limit of the sequence consisting, at the nth stage, of the largest quotient of the ordered group G_n in which every coordinate of x_n is equal to one is a non-zero quotient of G every prime quotient of which is $\mathbb{Z}$. As soon as Condition 2.1 (i) holds for x_n in G_n, then it holds for x in G. In other words, if G has no non-zero quotient isomorphic to $\mathbb{Z}$, then it has the property (D).

Proposition 10. *Let (G, G^+) be a dimension group with the property (D). Denote by G^{++} the set of all order units in G, and suppose that $G^{++} \neq \emptyset$. Then $(G, G^{++} \cup \{0\})$ is a simple dimension group.*

Proof. Observe first that $G^{++} + G^+ = G^{++}$. With this fact (and with the assumption that G^{++} is non-empty) it is straightforward to check that $(G, G^{++} \cup \{0\})$ is an ordered abelian group. We proceed to show that it is a dimension group. This ordered group is unperforated as (G, G^+) is, and so we need only show that it has the Riesz decomposition property. Equip G with the two orderings $\le$ and $\precsim$ given by $x \le y$ if $y - x \in G^+$ and $x \precsim y$ if $y - x \in G^{++} \cup \{0\}$. Suppose that $x \precsim y_1 + y_2$ where $x, y_1, y_2 \in G^{++} \cup \{0\}$. We must find $x_1, x_2 \in G^{++} \cup \{0\}$ such that $x = x_1 + x_2$ and $x_j \precsim y_j$, $j = 1, 2$. It is trivial to find x_1 and x_2 in the cases that one of x, y_1, y_2, and $y_1 + y_2 - x$ is zero. Suppose that the four elements above are non-zero, in which case by hypothesis they all are order units. By hypothesis (and by Lemma 8) there is $z \in G^{++}$ such that

$$2z \le x, \quad z \le y_1, \quad z \le y_2, \quad 2z \le y_1 + y_2 - x.$$

Then $x - 2z \le (y_1 - 2z) + (y_2 - 2z)$. Since (G, G^+) has the Riesz decomposition property there are $v_1, v_2 \in G^+$ such that

$$x - 2z = v_1 + v_2, \qquad v_1 \le y_1 - 2z, \qquad v_2 \le y_2 - 2z.$$

Set $v_1 + z = x_1$ and $v_2 + z = x_2$. Then x_1, x_2 belong to G^{++}, $x = x_1 + x_2$, $x_1 \precsim y_1$, and $x_2 \precsim y_2$; the latter two inequalities hold because

$$y_j - x_j \;=\; y_j - v_j - z \;=\; (y_j - v_j - 2z) + z \;\in\; G^+ + G^{++} \;=\; G^{++}.$$

□

Proposition 11. *Let A be a unital AF-algebra A with no non-zero finite-dimensional representation. There exists a unital sub-C*-algebra B of A which is a simple, infinite-dimensional AF-algebra, and for which the inclusion mapping $B \to A$ gives rise to*

1. *an isomorphism of simplices $T(A) \to T(B)$, and*
2. *an isomorphism of groups $K_0(B) \to K_0(A)$ which maps $K_0(B)^+$ onto $K_0(A)^{++} \cup \{0\}$, and so in particular,*

$$(K_0(B), K_0(B)^+, [1]) \cong (K_0(A), K_0(A)^{++} \cup \{0\}, [1]).$$

If A is an initial object, then so also is B.

Proof. We derive from Lemma 9 that $K_0(A)$ has property (D), and we then conclude from Proposition 10 that $K_0(A)$ equipped with the positive cone $G^+ := K_0(A)^{++} \cup \{0\}$ is a simple dimension group. Let B_1 be the simple, unital, infinite-dimensional AF-algebra with dimension group $(K_0(A), G^+, [1_A])$, and use the homomorphism theorem for AF-algebras ([12, Proposition 1.3.4 (iii)]), to find a unital (necessarily injective) *-homomorphism $\varphi\colon B_1 \to A$ which induces the (canonical) homomorphism $K_0(B_1) \to K_0(A)$ that maps $K_0(B_1)^+$ onto G^+ and $[1_{B_1}]$ onto $[1_A]$. Set $\varphi(B_1) = B$. Then B is a unital sub-C*-algebra of A, B is isomorphic to B_1, and (ii) holds.

The property (i) follows from (ii) and the fact, that we shall prove, that the state spaces of $(K_0(A), K_0(A)^+, [1_A])$ and $(K_0(A), G^+, [1_A])$ coincide. The former space is contained in the latter because G^+ is contained in $K_0(A)^+$. To show the reverse inclusion take a state f on $(K_0(A), G^+, [1_A])$ and take $g \in K_0(A)^+$. We must show that $f(g) \geq 0$. Use Lemmas 8 and 9 to find for each natural number n an element v_n in $K_0(A)^{++}$ such that $nv_n \leq [1_A]$. Then $nf(v_n) \leq 1$, so $f(v_n) \leq 1/n$; and $g + v_n$ belongs to $K_0(A)^{++}$, so $f(g + v_n) \geq 0$. These two inequalities, that hold for all n, imply that $f(g) \geq 0$. □

Corollary 12.

1. *There is a simple unital infinite-dimensional AF-algebra which is an initial object.*
2. *The Jiang-Su algebra $\mathcal{Z}$ is an initial object.*

Proof. The assertion (i) follows immediately from Propositions 6 and 11.

The assertion (ii) follows from (i) and the fact, proved in [7], that the Jiang-Su algebra $\mathcal{Z}$ embeds in (actually is tensorially absorbed by) any unital simple infinite-dimensional AF-algebra. □

The corollary above provides an affirmative answer to Question 3 (and hence also to Questions 1 and 2) of the introduction in the case that the target C*-algebra A is assumed to be of real rank zero.

The question of initial objects may perhaps be pertinent in the classification program, where properties such as approximate divisibility and being able to absorb the Jiang-Su algebra $\mathcal{Z}$ are of interest. We remind the reader that a C*-algebra A is approximately divisible if for each natural number n there is a sequence $\varphi_k \colon M_n \oplus M_{n+1} \to \mathcal{M}(A)$ of unital *-homomorphisms (where $\mathcal{M}(A)$ denotes the multiplier algebra of A) such that $[\varphi_k(x), a] \to 0$ for all $a \in A$ and all $x \in M_n \oplus M_{n+1}$. (It turns out that if A is unital, then we need only find such a sequence of *-homomorphisms for $n = 2$.) It is easily seen that a separable C*-algebra A is approximately divisible if, and only if, there is a unital *-homomorphism

$$\prod_{n\in\mathbb{N}} (M_n \oplus M_{n+1}) / \sum_{n\in\mathbb{N}} (M_n \oplus M_{n+1}) \to \mathcal{M}(A)_\omega \cap A', \tag{1}$$

and it follows from [12, Theorem 7.2.2] and [7] that A is $\mathcal{Z}$-absorbing if and only if there is a unital embedding of $\mathcal{Z}$ into $\mathcal{M}(A)_\omega \cap A'$; here, ω is any free ultrafilter on $\mathbb{N}$, and $\mathcal{M}(A)$ is identified with a sub-C*-algebra of the ultrapower $\mathcal{M}(A)_\omega$ (the C*-algebra of bounded sequences in $\mathcal{M}(A)$, modulo the ideal of bounded sequences convergent to 0 along ω).

Toms and Winter recently observed ([13]) that any separable approximately divisible C*-algebra is $\mathcal{Z}$-absorbing, because one can embed $\mathcal{Z}$ unitally into the C*-algebra on the left-hand side of (1). (The latter fact follows from our Corollary 12, but it can also be proved directly, as was done in [13].) In the general case, when A need not be approximately divisible, it is of interest to decide when A is $\mathcal{Z}$-absorbing, or, equivalently, when one can find a unital embedding of $\mathcal{Z}$ into $\mathcal{M}(A)_\omega \cap A'$. Here it would be extremely useful if one knew that $\mathcal{Z}$ was an initial object in the category of all unital C*-algebras with no non-zero finite-dimensional representations.

The proof of Corollary 12 yields an explicit—at the level of the invariant—simple unital AF-algebra which is an initial object. Indeed, consider the initial object $P = \bigotimes_{n=1}^{\infty} M_2 \oplus M_3$, the K_0-group of which is the dense subset G of $C(X, \mathbb{R})$ described above (after Corollary 7), with the relative order, where $C(X, \mathbb{R})$ is equipped with the standard pointwise ordering. The simple dimension group $(G, G^{++} \cup \{0\})$ of Proposition 10 is obtained by again viewing G as a subgroup of $C(X, \mathbb{R})$ but this time endowing $C(X, \mathbb{R})$ with the strict pointwise ordering (in which an element $f \in C(X, \mathbb{R})$ is positive if $f = 0$ or if $f(x) > 0$ for all $x \in X$). Any other simple dimension group which maps onto this may also be used.

It would of course be nice to have an even more explicit (or natural) example of a simple unital infinite-dimensional AF-algebra which is an initial object in the sense of this paper.

The trace simplex of the simple unital AF-algebra referred to above is the simplex of probability measures on the Cantor set. We shall show in Section 5 that the trace simplex of an initial object, that has sufficiently many projections, must be infinite-dimensional. Let us now note that a large class of infinite-dimensional Choquet simplices arise as the trace simplex of an initial object.

Proposition 13. *Let X be a metrizable compact Hausdorff space which admits an embedding of the Cantor set.*[6] *There exists a simple unital AF-algebra A which is an initial object, such that $T(A)$ is affinely homeomorphic to the simplex $\mathcal{M}_1(X)$ of (Borel) probability measures on X.*

Proof. By hypothesis X has a closed subset X_0 which is (homeomorphic to) the Cantor set. The dimension group of the known initial object $\bigotimes_{n=1}^{\infty} M_2 \oplus M_3$ is isomorphic in a natural way to a dense subgroup G of $C(X_0, \mathbb{R})$ (equipped with the standard pointwise ordering), with canonical order unit corresponding to the constant function 1_{X_0}, cf. the remark after Corollary 7. We shall construct below a countable dense subgroup H of $C(X, \mathbb{R})$ such that the constant function 1_X belongs to H, and such that the restriction $f|_{X_0}$ belongs to G for every $f \in H$. Equip H with the strict pointwise ordering on $C(X, \mathbb{R})$ and with the order unit 1_X. Then we have an ordered group homomorphism $\varphi\colon H \to G$ given by $\varphi(f) = f|_{X_0}$, which maps 1_X into 1_{X_0}. It follows that we may take A to be the unital, simple AF-algebra with invariant $(H, H^+, 1_X)$, as by the homomorphism theorem for AF-algebras (cf. above) φ induces a unital embedding of A into $\bigotimes_{n=1}^{\infty} M_2 \oplus M_3$, whence A is an initial object, and the trace simplex of A is homeomorphic to the state space of $(H, H^+, 1_X)$, which is $\mathcal{M}_1(X)$.

Let us now pass to the construction of H. Each $g \in G$ extends to $\tilde{g} \in C(X, \mathbb{R})$ (we do not make any assumption concerning the mapping $g \mapsto \tilde{g}$). Choose a countable dense subgroup H_0 of $C_0(X \setminus X_0, \mathbb{R}) \subseteq C(X, \mathbb{R})$, and consider the countable subgroup of $C(X, \mathbb{R})$ generated by H_0 and the countable set $\{\tilde{g} : g \in G\}$. Denote this group, with the relative (strict pointwise) order, by H; let us check that this choice of H fulfils the requirements. First, $f|_{X_0} \in G$ for every $f \in H$. To see that H is dense in $C(X, \mathbb{R})$, let there be given $f \in C(X, \mathbb{R})$ and $\varepsilon > 0$. Choose $g \in G$ such that $\|f|_{X_0} - g\|_\infty < \varepsilon/2$. Extend $f|_{X_0} - g$ to a function $f_0 \in C(X, \mathbb{R})$ with $\|f_0\|_\infty = \|f|_{X_0} - g\|_\infty < \varepsilon/2$. Note that $f - \tilde{g} - f_0$ belongs to $C_0(X \setminus X_0, \mathbb{R})$. Choose $h_0 \in H_0$ such that $\|f - \tilde{g} - f_0 - h_0\|_\infty < \varepsilon/2$, and consider the function $h = \tilde{g} + h_0 \in H$. We have $\|f - h\|_\infty \le \|f - \tilde{g} - f_0 - h_0\|_\infty + \|f_0\|_\infty < \varepsilon$, as desired. □

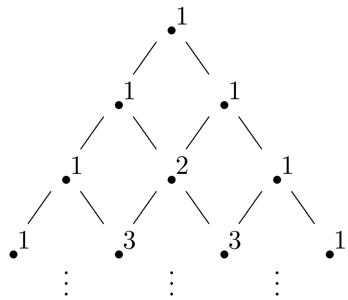
[6] An equivalent formulation of this (rather weak) property is that X has a non-empty closed subset with no isolated points.

4 Hausdorff Moments, the GICAR-Algebra, and Pascal's Triangle

In this section we shall establish the following result.

Theorem 14. *The GICAR-algebra is an initial object (in the sense of Definition 4).*

We review some of the background material. Consider the Bratteli diagram given by Pascal's triangle,



and denote by

$$\mathbb{C} = B_0 \to B_1 \to B_2 \to \cdots \to \varinjlim B_n \ (= B)$$

the inductive system of finite-dimensional C*-algebras associated with that Bratteli diagram. The C*-algebra B is the GICAR-algebra. (It can also, more naturally, be realized as the fixed point algebra of the CAR-algebra under a certain action of the circle referred to as the gauge action, cf. [3].)

For each $n \geq 0$ and $0 \leq k \leq n$, choose a minimal projection $e(n,k)$ in the kth minimal direct summand of B_n. Note that $e(0,0) = 1_B$ and that $e(n,k)$ is Murray-von Neumann equivalent to $e(n+1,k) + e(n+1,k+1)$ in B_{n+1}. A trace τ on B_n is determined by its values on the projections $e(n,k)$, $0 \leq k \leq n$.

The group $K_0(B)$ is generated, as an ordered abelian group, by the elements $[e(n,k)]$, with $n \geq 0$ and $0 \leq k \leq n$; that is, these elements span $K_0(B)$ as an abelian group, and the semigroup spanned by the elements $[e(n,k)]$ is equal to $K_0(B)^+$. Our generators satisfy the relations

$$[e(n,k)] = [e(n+1,k)] + [e(n+1,k+1)], \qquad n \geq 0, \ 0 \leq k \leq n. \tag{2}$$

Moreover, $(K_0(B), K_0(B)^+)$ is the universal ordered abelian group generated, as an ordered abelian group, by elements $g(n,k)$, $n \geq 0$ and $0 \leq k \leq n$, with the relations $g(n,k) = g(n+1,k) + g(n+1,k+1)$.

For brevity we shall set $(K_0(B), K_0(B)^+, [1_B]_0) = (H, H^+, v)$.

For each abelian (additively written) group G and for each sequence

$t\colon \mathbb{N}_0 \to G$ associate the discrete derivative $t'\colon \mathbb{N}_0 \to G$ given by $t'(k) = t(k) - t(k+1)$. Denote the nth derivative of t by $t^{(n)}$, and apply the convention $t^{(0)} = t$.

We remind the reader of the following classical result. The equivalence of (i) and (iv) is the solution to the Hausdorff Moment problem (see e.g. [1, Proposition 6.11]). The equivalence of (i), (ii), and (iii) follows from Proposition 16 below (with $(G, G^+, u) = (\mathbb{R}, \mathbb{R}^+, 1)$).

Proposition 15 (Hausdorff Moments). *The following four conditions are equivalent for any sequence* $t\colon \mathbb{N}_0 \to \mathbb{R}$.

(i) $t^{(k)}(n) \geq 0$ *for all* $n, k \geq 0$.
(ii) *There is a system,* $\{t(n,k)\}_{0\leq k\leq n}$*, of positive real numbers (necessarily unique) such that*

$$t(n+1,k) + t(n+1,k+1) = t(n,k), \qquad t(n,n) = t(n),$$

for $n \geq 0$ *and* $0 \leq k \leq n$.
(iii) *There is a (unique) tracial state* τ *on the GICAR-algebra such that* $t(n) = \tau(e(n,n))$ *for all* $n \geq 0$.
(iv) *There is a Borel probability measure* μ *on the interval* $[0,1]$ *such that*

$$t(n) = \int_0^1 \lambda^n \, d\mu(\lambda),$$

for all $n \geq 0$.

It follows from Proposition 16 below and from (iv) that the coefficients $t(n,k)$ from (ii) are given by

$$t(n,k) = t^{(n-k)}(k) = \int_0^1 \lambda^k (1-\lambda)^{n-k} \, d\mu(\lambda). \tag{3}$$

A sequence $t = (t(0), t(1), \dots)$ satisfying the condition in Proposition 15 (iv) (or, equivalently, the three other conditions of Proposition 15) is called a *Hausdorff moment sequence*. Note that $t(0) = 1$ in every Hausdorff moment sequence. Let us denote the set of all moment sequences by $\mathcal{M}$. Note that $\mathcal{M}$ is a compact convex set and in fact a Choquet simplex. For each $n \in \mathbb{N}_0$ let us set

$$\mathcal{M}_n = \big\{ \big(t(0), t(1), t(2), \dots, t(n)\big) : \big(t(0), t(1), t(2), \dots\big) \in \mathcal{M} \big\} \subseteq \mathbb{R}^{n+1},$$

and denote by π_n the canonical surjective affine mapping $\mathcal{M}_{n+1} \to \mathcal{M}_n$.

Let us say that a moment sequence $t = (t(0), t(1), t(2), \dots)$ is *trivial* if the corresponding measure in Proposition 15 (iv) is supported in $\{0,1\}$, and say that t is *non-trivial* otherwise. A sequence t is trivial if and only if it is a convex combination of the two trivial sequences $(1,1,1,\dots)$ and $(1,0,0,\dots)$.

It follows from this and (iv) above that t is non-trivial if and only if $t(2) < t(1)$. One can use Equation (3) to see that t is non-trivial if and only if $t(n,k) \neq 0$ for all n and k.

We seek unital embeddings from the GICAR algebra B into unital AF-algebras (and into unital C*-algebras of real rank zero). At the level of the invariant we are thus seeking positive unit preserving group homomorphisms from the dimension group with distinguished unit (H, H^+, v) associated to the GICAR algebra into the ordered K_0-group with distinguished unit of the target algebra; call this invariant (G, G^+, u). The proposition below rephrases this problem as that of the existence of a function $g\colon \mathbb{N}_0 \to G$ with certain properties.

Proposition 16. *Let (H, H^+, v) be as above, and let (G, G^+, u) be an ordered abelian group with a distinguished order unit u. Let $g\colon \mathbb{N}_0 \to G$ be given, and assume that $g(0) = v$. The following conditions are equivalent.*

1. *$g^{(k)}(n) \in G^+$ for all $n, k \geq 0$.*
2. *There is a system, $\{g(n,k)\}_{0 \leq k \leq n}$, of elements in G^+ (necessarily unique) such that*
$$g(n+1,k) + g(n+1,k+1) = g(n,k), \qquad g(n,n) = g(n),$$
for all $n \geq 0$ and $0 \leq k \leq n$.
3. *There is a (unique) homomorphism of ordered groups $\varphi\colon H \to G$ with $\varphi(v) = u$ such that $\varphi([e(n,n)]) = g(n)$ for all $n \geq 0$.*

If the three conditions above are satisfied, then

$$\varphi([e(n,k)]) = g(n,k) = g^{(n-k)}(k)$$

for all $n \geq 0$ and $0 \leq k \leq n$; and the homomorphism φ is faithful if and only if $g(n,k)$ is non-zero for all $n \geq 0$ and $0 \leq k \leq n$.

Proof. (i) $\Rightarrow$ (ii). Set $g(n,k) = g^{(n-k)}(k) \in G^+$. Then $g(n,n) = g^{(0)}(n) = g(n)$, and

$$\begin{aligned} g(n,k) - g(n+1,k+1) = g^{(n-k)}(k) - g^{(n-k)}(k+1) \;&=\; g^{(n-k+1)}(k) \\ = g(n+1,k). \end{aligned}$$

(ii) $\Rightarrow$ (iii). We noted after Theorem 14 that $H = K_0(B)$ is generated, as an ordered abelian group, by the elements $[e(n,k)]$, $n \geq 0$ and $0 \leq k \leq n$, and that H is the universal ordered abelian group generated by these elements subject to the relations (2). Accordingly, by (ii), there exists a (unique) positive group homomorphism $\varphi\colon H \to G$ with $\varphi([e(n,k)]) = g(n,k)$. Also, $\varphi(v) = \varphi([e(0,0)]) = g(0,0) = g(0) = u$.

To complete the proof we must show that φ is uniquely determined by its value on the elements $[e(n,n)]$, $n \geq 0$. But this follows from the fact that the elements $[e(n,k)]$, with $n \geq 0$ and $0 \leq k \leq n$, belong to the subgroup generated by the elements $[e(n,n)]$, for $n \geq 0$, by the relations (2).

(iii) $\Rightarrow$ (i). This implication follows from the identity $\varphi([e(n+k,n)]) = g^{(k)}(n)$, that we shall proceed to prove by induction on k. The case $k=0$ is explicitly contained in (iii). Assume that the identity has been shown to hold for some $k \geq 0$. Then, by (2),

$$\begin{aligned} g^{(k+1)}(n) &= g^{(k)}(n) - g^{(k)}(n+1) = \varphi\big([e(n+k,n)] - [e(n+k+1,n+1)]\big) \\ &= \varphi([e(n+k+1,n)]). \end{aligned}$$

To prove the two last claims of the proposition, assume that g satisfies the three equivalent conditions, and consider the homomorphism of ordered groups $\varphi\colon H \to G$ asserted to exist in (iii). It follows from the proofs of (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) that $\varphi([e(n,k)]) = g(n,k) = g^{(n-k)}(k)$. Any non-zero positive element h of H is a finite (non-empty) sum of elements of the form $[e(n,k)]$. Thus $\varphi(h)$ is a finite (non-empty) sum of elements of the form $g(n,k)$. This shows that $\varphi(h)$ is non-zero for all non-zero positive elements h in H if and only if $g(n,k)$ is non-zero for all n and k. □

Let us now return to the convex set $\mathcal{M}$ of Hausdorff moment sequences in $\mathbb{R}^+$ and to the truncated finite-dimensional convex sets $\mathcal{M}_n$.

Lemma 17. $\dim(\mathcal{M}_n) = n$.

Proof. The convex set $\mathcal{M}_n$ is a subset of $\{1\} \times \mathbb{R}^n$, and has therefore dimension at most n. On the other hand, the points $(1, \lambda, \lambda^2, \dots, \lambda^n)$ belong to $\mathcal{M}_n$ for each $\lambda \in (0,1)$, and these points span an n-dimensional convex set. □

Let $\mathcal{M}_n^\circ$ denote the relative interior[7] of $\mathcal{M}_n$. By standard theory for finite-dimensional convex sets, see e.g. [4, Theorem 3.4], $\dim(\mathcal{M}_n^\circ) = \dim(\mathcal{M}_n) = n$. Note that

$$\mathcal{M}_1 = \{(1,\lambda) : \lambda \in [0,1]\}, \qquad \mathcal{M}_1^\circ = \{(1,\lambda) : \lambda \in (0,1)\}.$$

For $n \geq 2$ we can use Lemma 17 to conclude that $\mathcal{M}_n^\circ = \{1\} \times U_n$ for some open convex subset U_n of $\mathbb{R}^n$.

Lemma 18. $\pi_n(\mathcal{M}_{n+1}^\circ) = \mathcal{M}_n^\circ$.

Proof. This follows from the standard fact from the theory for finite-dimensional convex sets (see e.g. [4, §3 and Exercise 3.3]) that the relative interior of the image of π_n is the image under π_n of the relative interior of $\mathcal{M}_{n+1}$ (combined with the fact that π_n is surjective). □

[7] The relative interior of a finite-dimensional convex set is its interior relatively to the affine set it generates.

Theorem 19. *Let G be a dense subset of the reals that contains 1. Then there is a non-trivial moment sequence $(t_0, t_1, t_2, \dots)$ such that t_n belongs to G for every $n \in \mathbb{N}_0$. Furthermore, the moment sequences with all terms belonging to G constitute a dense*[8] *subset of $\mathcal{M}$. If G also is a group, and has infinite rank over $\mathbb{Q}$, then there exists a moment sequence in G the terms of which are independent over $\mathbb{Q}$.*

Proof. Let $(s_0, s_1, s_2, \dots)$ be a moment sequence, let m be a natural number, and let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m$ be strictly positive real numbers. Since $(s_0, s_1, \dots, s_m)$ belongs to $\mathcal{M}_m$, since $\mathcal{M}_m^\circ$ is dense in $\mathcal{M}_m$ (cf. [4, Theorem 3.4]) and is equal to $\{1\} \times U_m$ for some open subset U_m of $\mathbb{R}^m$, since $1 \in G$, and since G is dense in $\mathbb{R}$, we can find $(t_0, t_1, \dots, t_m)$ in $\mathcal{M}_m^\circ$ such that t_j belongs to G for $j = 0, 1, \dots, m$ and $|t_j - s_j| < \varepsilon_j$ for $j = 1, \dots, m$.

Let us choose inductively t_n, $n > m$, such that $t_n \in G$ and $(t_0, t_1, \dots, t_n) \in \mathcal{M}_n^\circ$. Supppose that $n \geq m$ and that $t_0, t_1, \dots, t_n$ have been found. The set

$$\{s \in \mathbb{R} : (t_0, t_1, \dots, t_n, s) \in \mathcal{M}_{n+1}^\circ\}$$

is non-empty (by Lemma 18) and open (because $\mathcal{M}_{n+1}^\circ = \{1\} \times U_{n+1}$ for some open subset U_{n+1} of $\mathbb{R}^{n+1}$). Hence there exists $t_{n+1} \in G$ such that $(t_0, t_1, \dots, t_{n+1}) \in \mathcal{M}_{n+1}^\circ$.

The resulting sequence $(t_0, t_1, t_2, \dots)$ in G is a moment sequence by construction and is close to the given moment sequence $(s_0, s_1, s_2, \dots)$.

The inequality $t_2 < t_1$ holds because (t_0, t_1, t_2) belongs to the open set $\mathcal{M}_2^\circ = \{1\} \times U_2$. (Indeed, note that $t_1 \leq t_2$ whenever (t_0, t_1, t_2) belongs to $\mathcal{M}_2$ and, hence, that the element $(1, t_1, t_1)$ of $\mathcal{M}_2$ belongs to the boundary.)

Concerning the desired independence of the terms of the moment sequence when G is a group, of infinite rank, it will suffice to choose each t_n in the set

$$G \setminus \operatorname{span}_{\mathbb{Q}}\{t_0, t_1, \dots, t_{n-1}\}.$$

This is possible because this set is dense in $\mathbb{R}$ by the assumption on G. □

Corollary 20. *Let G be a dense subgroup of $\mathbb{R}$ with $1 \in G$. There is a faithful homomorphism of ordered groups from the dimension group H associated with the Pascal triangle to G (with the order inherited from $\mathbb{R}$) that maps the canonical order unit of H to 1. Furthermore, the set of such maps into G is dense in the set of such maps just into $\mathbb{R}$, in the topology of pointwise convergence on H. If G is of infinite rank there exists such a map which is injective.*

Proof. Propositions 15 and 16 give a one-to-one correspondence between moment sequences $t \colon \mathbb{N}_0 \to G$ and homomorphisms $\varphi \colon H \to G$ of ordered abelian groups that map the canonical order unit $v \in H$ into $1 \in G$, such that $\varphi([e(n,k)]) = t(n,k)$ for all $n \geq 0$ and $0 \leq k \leq n$. If t is non-trivial, then

[8] In the standard pointwise (or product) topology.

$t(n,k)$ is non-zero for all n, k, whence $\varphi(g) > 0$ for every non-zero positive element g in H (because each such element g is a sum of elements of the form $[e(n,k)]$).

A pointwise converging net of moment sequences corresponds to a pointwise converging net of homomorphisms $H \to G$.

The first two claims now follow from Theorem 19.

A homomorphism $\varphi\colon H \to G$ is injective if the restriction of φ to the subgroup spanned by $\{[e(n,k)] : k = 0, 1, \dots, n\}$ is injective for every n. The latter holds, for a specific n, if and only if $t(n,0), t(n,1), \dots, t(n,n)$ are independent over $\mathbb{Q}$, or, equivalently, if and only if $t(0), t(1), \dots, t(n)$ are independent over $\mathbb{Q}$. (Use the relation in Proposition 15 (ii) to see the second equivalence.) This shows that a moment sequence $t\colon \mathbb{N}_0 \to G$, where $t(0), t(1), \dots$ are independent over $\mathbb{Q}$, gives rise to an injective homomorphism $\varphi\colon H \to G$. The existence of such a moment sequence t, under the assumption that G has infinite rank, follows from Theorem 19. □

Lemma 21. *With X the Cantor set, let $f_1, \dots, f_n\colon X \to \mathbb{R}$ be continuous functions, and let $U \subseteq \mathbb{R}^{n+1}$ be an open subset such that*

$$\{s \in \mathbb{R} : (f_1(x), f_2(x), \dots, f_n(x), s) \in U\}$$

is non-empty for every $x \in X$. It follows that there exists a continuous function $f_{n+1}\colon X \to \mathbb{R}$ such that

$$(f_1(x), f_2(x), \dots, f_n(x), f_{n+1}(x)) \in U$$

for all $x \in X$.

Proof. For each $s \in \mathbb{R}$ consider the set V_s of those $x \in X$ for which $(f_1(x), f_2(x), \dots, f_n(x), s)$ belongs to U. Then $(V_s)_{s\in\mathbb{R}}$ is an open cover of X, and so by compactness, X has a finite subcover $V_{s_1}, V_{s_2}, \dots, V_{s_k}$. Because X is totally disconnected there are clopen subsets $W_j \subseteq V_{s_j}$ such that $W_1, W_2, \dots, W_k$ partition X. The function $f_{n+1} = \sum_{j=1}^{k} s_j 1_{W_j}$ is as desired. □

Proposition 22. *With X the Cantor set, let G be a norm-dense subset of $C(X, [0,1])$ that contains the constant function 1. There exists a sequence $(g_0, g_1, g_2, \dots)$ in G such that $(g_0(x), g_1(x), g_2(x), \dots)$ is a non-trivial moment sequence for every $x \in X$.*

Proof. Choose $g_0, g_1, \dots$ in G inductively such that $(g_0(x), g_1(x), \dots, g_n(x))$ belongs to $\mathcal{M}_n^\circ$ for every $x \in X$. Begin by choosing g_0 to be the constant function 1 (as it must be). Suppose that $n \geq 0$ and that $g_0, g_1, \dots, g_n$ as above have been found. As observed earlier, $\mathcal{M}_{n+1}^\circ = \{1\} \times U_{n+1}$ for some open subset U_{n+1} of $\mathbb{R}^{n+1}$. The set

$$\begin{aligned} &\{s \in \mathbb{R} : (g_1(x), \dots, g_n(x), s) \in U_{n+1}\} \\ &= \{s \in \mathbb{R} : (g_0(x), g_1(x), \dots, g_n(x), s) \in \mathcal{M}_{n+1}^\circ\} \end{aligned}$$

is non-empty for each $x \in X$ (by Lemma 18), and so we can use Lemma 21 to find a continuous function $f\colon X \to \mathbb{R}$ such that $(g_1(x), \dots, g_n(x), f(x))$ belongs to U_{n+1} for all $x \in X$. By compactness of X, continuity of the functions $g_1, \dots, g_n, f$, and because U_{n+1} is open, there exists $\delta > 0$ such that $(g_1(x), \dots, g_n(x), h(x))$ belongs to U_{n+1} for all $x \in X$ whenever $\|f - h\|_\infty < \delta$. As G is dense in $C(X, \mathbb{R})$ we can find $g_{n+1} \in G$ with $\|f - g_{n+1}\|_\infty < \delta$, and this function has the desired properties.

As in the proof of Proposition 15, since $(g_0(x), g_1(x), g_2(x))$ belongs to $\mathcal{M}_2^\circ$, we get $g_2(x) < g_1(x)$, which in turns implies that the moment sequence $(g_0(x), g_1(x), g_2(x), \dots)$ is non-trivial for every $x \in X$. □

Proposition 23. *With X the Cantor set, let G be a norm-dense subgroup of $C(X, \mathbb{R})$ that contains the constant function 1. There exists a faithful homomorphism of ordered groups from the dimension group H associated with the Pascal triangle to G (with the strict pointwise order) that takes the distinguished order unit v of H into the constant function 1.*

Proof. Choose a sequence $g_0, g_1, g_2, \dots$ in G as specified in Proposition 22, and consider the (unique) system $\{g(n,k)\}_{0 \le k \le n}$ in G such that

$$g(n+1,k) + g(n+1,k+1) = g(n,k), \qquad g(n,n) = g_n$$

for $n \ge 0$ and $0 \le k \le n$. Use Proposition 15 and the non-triviality of the moment sequence $(g_0(x), g_1(x), g_2(x), \dots)$ to conclude that $g(n,k)(x) > 0$ for all $x \in X$. Hence, by Proposition 16, there exists a homomorphism of ordered groups $\varphi\colon H \to G$ such that $\varphi([e(n,k)]) = g(n,k)$ for all $n \ge 0$ and $0 \le k \le n$.

Each function $g(n,k)$ is strictly positive, and hence non-zero, so it follows from Proposition 16 that φ is faithful. □

Proof of Theorem 14. By Corollary 7 it suffices to find a unital embedding of the GICAR-algebra B into the AF-algebra $P = \bigotimes_{n=1}^\infty M_2 \oplus M_3$. The ordered K_0-group of P is (isomorphic to) a dense subgroup G of $C(X, \mathbb{R})$ which contains the constant function 1 (as shown immediately after Corollary 7). The existence of a unital embedding of the GICAR-algebra into the AF-algebra P now follows from Proposition 23. □

5 Properties of Initial Objects

We shall show in this last section that initial objects in the sense of this paper, although abundant, form at the same time a rather special class of C*-algebras.

An element g in an abelian group G will be said to be *infinitely divisible* if the set of natural numbers n for which the equation $nh = g$ has a solution $h \in G$ is unbounded.

Proposition 24. *If B is an initial object, then $K_0(B)^+$ contains no non-zero infinitely divisible elements.*

Proof. There exists a unital C*-algebra A of real rank zero and with no non-zero finite-dimensional representations, such that no non-zero element in $K_0(A)$ is infinitely divisible, and such that any non-zero projection has a non-zero class in $K_0(A)$. (For example, any irrational rotation C*-algebra.) If B is an initial object, then B embeds into A, and by choice of A the corresponding ordered group homomorphism $K_0(B) \to K_0(A)$ takes any non-zero positive element of $K_0(B)$ into a non-zero positive element of $K_0(A)$. Since the image of an infinitely divisible element is again infinitely divisible, no non-zero element of $K_0(B)^+$ can be infinitely divisible. □

Lemma 25. *Let (G, G^+, u) be an ordered abelian group with order unit. Let $p_1, p_2, \ldots, p_n$ be distinct primes and suppose that $f_1, \ldots, f_n$ are states on (G, G^+, u) such that $f_j(G) = \mathbb{Z}[1/p_j]$ for $j = 1, \ldots, n$. Then $f_1, \ldots, f_n$ are affinely independent.*

Proof. The assertion is proved by induction on n. It suffices to show that for every natural number n, for every set of distinct primes $p_1, \ldots, p_n, q$, and for every set of states $f_1, \ldots, f_n, f$ on (G, G^+, u), with $f_j(G) = \mathbb{Z}[1/p_j]$ and $f(G) = \mathbb{Z}[1/q]$ and with $f_1, \ldots, f_n$ affinely independent, f is not an affine combination of $f_1, \ldots, f_n$.

Suppose, to reach a contradiction, that $f = \alpha_1 f_1 + \cdots + \alpha_n f_n$, with $\alpha_1, \ldots, \alpha_n$ real numbers with sum 1. If $n = 1$, then $f = f_1$, which clearly is impossible. Consider the case $n \geq 2$. Since $f_1, \ldots, f_n$ are assumed to be affinely independent, there are $g_1, \ldots, g_{n-1} \in G$ such that the vectors

$$x_j = (f_j(g_1), f_j(g_2), \ldots, f_j(g_{n-1})) \in \mathbb{Q}^{n-1}, \qquad j = 1, 2, \ldots, n,$$

are affinely independent. The coefficients α_j above therefore constitute the unique solution to the equations

$$\begin{gathered} f_1(g_j)\alpha_1 + f_2(g_j)\alpha_2 + \cdots + f_n(g_j)\alpha_n = f(g_j), \qquad j = 1, 2, \ldots, n-1, \\ \alpha_1 + \cdots + \alpha_n = 1. \end{gathered}$$

As these n equations in the n unknowns $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent, and all the coefficients are rational, also $\alpha_1, \alpha_2, \ldots, \alpha_n$ must be rational.

Denote by $\mathcal{Q}'(q)$ the ring of all rational numbers with denominator (in reduced form) not divisible by q. Observe that $f_j(g) \in \mathcal{Q}'(q)$ for all $j = 1, \ldots, n$ and for all $g \in G$. There is a natural number k such that $q^k \alpha_j \in \mathcal{Q}'(q)$ for all $j = 1, \ldots, n$. Then

$$q^k f(g) = q^k \alpha_1 f_1(g) + \cdots + q^k \alpha_n f_n(g) \in \mathcal{Q}'(q),$$

for all $g \in G$. But this is impossible as, by hypothesis, $f(g) = 1/q^{k+1}$ for some $g \in G$. □

Proposition 26. *Let B be an initial object (in the sense of Definition 4), and suppose that no quotient of B has a minimal non-zero projection. Then the trace simplex $T(B)$ of B is necessarily infinite-dimensional.*

It follows in particular that any simple unital C*-algebra of real rank zero, other than $\mathbb{C}$, which is an initial object has infinite-dimensional trace simplex. (Note for this that no matrix algebra M_n with $n \geq 2$ is an initial object.)

Proof. Any initial object embeds by definition into a large class of C*-algebras that includes exact C*-algebras (such as for example any UHF-algebra), and is therefore itself exact, being a sub-C*-algebra of an exact C*-algebra (see [8, Proposition 7.1]). It follows (from [2] and [5], or from [6]) that the canonical affine map from the trace simplex $T(B)$ to the state space of $(K_0(B), K_0(B)^+, [1])$ is surjective. It is therefore sufficient to show that the latter space is infinite-dimensional. For each prime p there is a unital embedding of B into the UHF-algebra of type p^∞, and hence a homomorphism of ordered groups $f_p\colon K_0(B) \to \mathbb{Z}[1/p]$ with $f_p([1]) = 1$. Let us show that the homomorphisms f_p, when considered as states (i.e., homomorphisms of ordered groups with order unit from $(K_0(A), [1])$ to $(\mathbb{R}, 1)$), are affinely independent.

For each prime number p, the image of f_p is a subgroup of $\mathbb{Z}[1/p]$ which contains 1, but the only such subgroups are $\mathbb{Z}[1/p]$ itself and the subgroups $p^{-k}\mathbb{Z}$ for some $k \geq 0$. The latter cannot be the image of f_p because the image of B in our UHF-algebra, being isomorphic to a quotient of B, is assumed to have no minimal non-zero projections. (Indeed, if $\{p_n\}$ is a strictly decreasing sequence of projections in the sub-algebra of the UHF-algebra, and if τ is the tracial state on the UHF-algebra, then $\{\tau(p_n - p_{n+1})\}$ is a sequence of strictly positive real numbers which converges to 0.)

Hence $f_p(K_0(B)) = \mathbb{Z}[1/p]$ for each prime p. It now follows from Lemma 25 that the states $\{f_p : p \text{ prime}\}$ are affinely independent. This shows that the state space of $(K_0(B), K_0(B)^+, [1])$ is infinite-dimensional, as desired. □

We end our paper by raising the following question:

Problem 27. Characterise initial objects (in the sense of Definition 4) among (simple) unital AF-algebras.

We could of course extend the problem above to include all (simple) real rank zero C*-algebras, but we expect a nice(r) answer when we restrict our attention to AF-algebras. Propositions 24 and 26 give necessary, but not sufficient, conditions for being an initial object. (A simple AF-algebra that satisfies the conditions of Propositions 24 and 26 can contain a unital simple sub-AF-algebra that does not satisfy the condition in Proposition 26, and hence is not an initial object.)

References

1. C. Berg, J. P. R. Christensen, and P. Ressel, *Harmonic analysis on semigroups. Theory of positive definite and related functions*, Springer Verlag, New York, 1984.
2. B. Blackadar and M. Rørdam, *Extending states on preordered semigroups and the existence of quasitraces on C^*-algebras*, J. Algebra **152** (1992), 240–247.
3. O. Bratteli, *Crossed products of UHF-algebras by product type actions*, Duke Math. J. **46** (1979), 1–23.
4. A. Brøndsted, *An introduction to convex polytopes*, Graduate Texts in Mathematics, vol. 90, Springer Verlag, New York, 1983.
5. U. Haagerup, *Every quasi-trace on an exact C^*-algebra is a trace*, preprint, 1991.
6. U. Haagerup and S. Thorbjørnsen, *Random matrices and K-theory for exact C^*-algebras*, Documenta Math. **4** (1999), 341–450.
7. X. Jiang and H. Su, *On a simple unital projectionless C^*-algebra*, American J. Math. **121** (1999), no. 2, 359–413.
8. E. Kirchberg, *Commutants of unitaries in UHF-algebras and functorial properties of exactness*, J. Reine Angew. Math. **452** (1994), 39–77.
9. E. Kirchberg and M. Rørdam, *Non-simple purely infinite C^*-algebras*, American J. Math. **122** (2000), 637–666.
10. E. Kirchberg and M. Rørdam, *Infinite non-simple C^*-algebras: absorbing the Cuntz algebra $\mathcal{O}_\infty$*, Advances in Math. **167** (2002), no. 2, 195–264.
11. F. Perera and M. Rørdam, *AF-embeddings into C^*-algebras of real rank zero*, J. Funct. Anal. **217** (2004), no. 1, 142–170.
12. M. Rørdam, *Classification of Nuclear, Simple C^*-algebras*, Classification of Nuclear C^*-Algebras. Entropy in Operator Algebras (J. Cuntz and V. Jones, eds.), vol. 126, Encyclopaedia of Mathematical Sciences. Subseries: Operator Algebras and Non-commutative Geometry, no. VII, Springer Verlag, Berlin, Heidelberg, 2001, pp. 1–145.
13. A. Toms and W. Winter, *Strongly self-absorbing C^*-algebras*, preprint.
14. S. Zhang, *Matricial structure and homotopy type of simple C^*-algebras with real rank zero*, J. Operator Theory **26** (1991), no. 2, 283–312.

Twisted K-Theory and Modular Invariants: I Quantum Doubles of Finite Groups

David E Evans

School of Mathematics, Cardiff University,
Senghennydd Road, Cardiff CF24 4YG, Wales, UK.
EvansDE@cf.ac.uk

Summary. A twisted vector-bundle approach to α-induction and modular invariants.

1 Introduction and Statement of Results

The Verlinde algebra is central to conformal field theory and consequently also to the braided subfactor approach to modular invariants. In the braided subfactor approach to modular invariants one has first a factor N, which we can take to be type III, and a non-degenerately braided system of endomorphisms ${}_N\mathcal{X}_N$ of N whose fusion rules as sectors are precisely those of our Verlinde algebra.

Fixing a braided system of endomorphisms on a type III factor N, we look for inclusions $\iota : N \hookrightarrow M$ such that its dual canonical endomorphism $\theta = \bar{\iota}\iota$ decomposes as a sum of endomorphisms from ${}_N\mathcal{X}_N$. To produce a modular invariant from such an inclusion, we first employ the Longo-Rehren $\alpha^{\pm}$-induction method [48] of extending endomorphisms of N to those in M and then compute the dimensions of the intertwining spaces $Z_{\lambda,\mu} := \langle \alpha_\lambda^+, \alpha_\mu^- \rangle$. The matrix $Z_{N\subset M} = [Z_{\lambda,\mu}]$ thus constructed from a braided inclusion $N \subset M$ is a modular invariant [11, 21]. Now we use α-induction and the inclusion map ι to construct finite systems whose general theory has been developed in [11, 12]. Let us choose representative endomorphisms of each irreducible subsector of sectors of the form $[\iota\lambda\bar{\iota}]$, $\lambda \in {}_N\mathcal{X}_N$. Any subsector of $[\alpha_\lambda^+ \alpha_\mu^-]$ is automatically a subsector of $[\iota\nu\bar{\iota}]$ for some ν in ${}_N\mathcal{X}_N$ and since we assume the non-degeneracy of the braiding the converse also holds [11]. This set of sectors yields a system ${}_M\mathcal{X}_M$ of sectors in general non-commutative (the original sectors from the system ${}_N\mathcal{X}_N$ is commutative since it is braided). We define in a similar fashion the chiral systems ${}_M\mathcal{X}_M^{\pm}$ to be the subsystems of $\beta \in {}_M\mathcal{X}_M$ such that $[\beta]$ is an irreducible subsector of $[\alpha_\lambda^{\pm}]$. The neutral system ${}_M\mathcal{X}_M^0$ is defined as the intersection ${}_M\mathcal{X}_M^- \cap {}_M\mathcal{X}_M^+$ and is non-degenerately braided, so that we obtain ${}_M\mathcal{X}_M^0 \subset {}_M\mathcal{X}_M^{\pm} \subset {}_M\mathcal{X}_M$ (see e.g [8]). A braided subfactor

$N \subset M$ producing a modular invariant Z is said to be type I when the dual canonical endomorphism is determined by the vacuum row or column $\oplus Z_{0,\lambda}[\lambda] = \oplus Z_{\lambda,0}[\lambda]$. In this case, which is equivalent to chiral locality, we can identify both ${}_M\mathcal{X}_M^{\pm}$ with ${}_N\mathcal{X}_M$ (by $\beta \mapsto \beta \circ \iota$, $\beta \in {}_M\mathcal{X}_M^{\pm}$ if ι is the inclusion of $N \subset M$.)

There are two cases of interest where there are natural constructions of braided systems or Verlinde algebras. The first is the case of affine Lie algebras or loop groups and their positive energy representations. In this WZW or loop group settings, the modular data (S, and T matrices etc) can be constructed from representation theory of unitary integrable highest weight modules over affine Lie algebras or in exponentiated form from the positive energy representations of loop groups. The subfactor machinery is invoked as follows. Let LG be a loop group (associated to a simple, simply connected loop group G). Let $L_I G$ denote the subgroup of loops which are trivial off some proper interval $I \subset \mathbb{T}$. Then in each level k vacuum representation π_0 of LG, we naturally obtain a net of type III factors $\{N(I)\}$ indexed by proper intervals $I \subset \mathbb{T}$ by taking $N(I) = \pi_0(L_I G)''$ (see [59, 35, 6]). Since the Doplicher-Haag-Roberts DHR selection criterion is met in the (level k) positive energy representations π_λ, there are DHR endomorphisms λ naturally associated with them. The rational conformal field theory RCFT modular data matches that in the subfactor setting – the RCFT Verlinde fusion coincides with the (DHR superselection) sector fusion, i.e. that $N_{\lambda,\mu}^{\nu} = \langle \lambda\mu, \nu \rangle$. The statistics S- and T-matrices are identical with the Kac-Peterson S- and T-modular matrices which perform the conformal character transformations.

The second is the case of quantum double of finite groups. A given finite system of endomorphisms may not be commutative or even braided but by taking the subfactor analogue of the quantum double of Drinfeld we obtain a subfactor with a non-degenerately braided system of endomorphisms. This construction can in particular be applied to a finite group G. This *quantum double subfactor* is basically the same as the Longo-Rehren inclusion [48] and is a way of yielding braided systems from not necessarily commutative systems. The modular data from a quantum double subfactor was first established by Ocneanu [23, Section 12.6] using topological insight, later by Izumi [37] with an algebraic flavour (see also [55, 43]).

Twisted equivariant K-theory [4] is relevant for both of these settings. Here the equivariant K-theory is twisted by an element of $H^4(BG, \mathbb{Z})$. When G is a compact simply connected Lie group, this manifests itself through the equivariant cohomology group $H_G^3(G, \mathbb{Z})$ [3, 32, 29, 30, 31], and for a finite group through $H^3(G, \mathbb{T})$ [28, 33].

The quantum double of the finite group subfactor $M_0 \subset M_0 \rtimes G$, (where the finite group G acts outerly on a type III factor M_0) was identified by Ocneanu and later by Izumi (see [23]) to be the group-subgroup subfactor $N = M_0 \rtimes \Delta(G) \subset M_0 \rtimes (G \times G) = M$ where $\Delta(G) = \{(g, g) : g \in G\}$ denotes the diagonal subgroup of $G \times G$. This data can be twisted for every

$[\omega] \in H^3(G, \mathbb{T})$ [16], and the subfactor interpretation of this data is in [37]. The parameter $[\omega]$ is regarded as the level in this setting [16].

It is therefore natural to think in terms of $\Delta(G) \subset G \times G$ and in particular the ambient group $G \times G$ with the diagonal actions. Indeed according to [50], the module categories of the double are given by pairs (H, ψ) for H a subgroup of $G \times G$ and ψ an arbitrary element of the 2-cohomology group $H^2(H, \mathbb{T})$. The corresponding Frobenius algebra or Q-system gives rise to a subfactor $N \subset M$ and hence by [11, 20] produces a modular invariant through the α-induction machinery. The ${}_N\mathcal{X}_M$ sectors are identified with the Δ-H bundles or the the equivariant K-group $K^0_{\Delta \times H}(G \times G)$, and the ${}_M\mathcal{X}_M$ sectors with the H-H bundles or the equivariant K- group $K^0_{H \times H}(G \times G)$. These identifications of sectors with bundles is compatible with the natural product of sectors and the product of bundles mentioned above.

To translate between equivariant twisted bundles on G, for the adjoint action, and equivariant twisted bundles on $G \times G$ with the diagonal action on left and right we need a corresponding cocycle on $G \times G$. If ω is a 3-cocycle in $Z^3(G, \mathbb{T})$, we define the 3-cocycle $\alpha = \pi_1^*\omega - \pi_2^*\omega$ on $G \times G$ if π_1, π_2 are the projections of $G \times G$ on the first and second factors respectively.

In Sect. 2.5, we make precise the relationship between ${}^\alpha K^0_{G \times G}(G \times G)$ and ${}^\omega K^0_G(G)$ where $G \times G$ acts on $G \times G$ by the first factor acting on the left and the second on the right using the diagonal embedding, and G acts G by the adjoint action. The map $(a, b) \to ab^{-1}$ takes the G-G action to the adjoint G action. This identifies the two K-theories

This work begins the study of understanding α-induction and the subfactor approach to modular invariants through twisted equivariant K-theory for the case of quantum doubles of finite groups. This has been thoroughly analysed in [25] from the subfactor viewpoint and in [50] from the viewpoint of module categories. From this we should understand the modular invariants which can be realised by subfactors as arising from a subgroup H of $G \times G$ and possible 2-cohomology from $H^2(H, \mathbb{T})$. The corresponding full system will be the twisted equivariant K-theory $K^0_{H \times H}(G \times G)$, where H acts on the left and right in the natural way. We should identify two homomorphisms $\alpha^\pm$

$$K^0_{\Delta \times \Delta}(G \times G) \to K^0_{H \times H}(G \times G) \tag{1}$$

whose images commute and generate $K^0_{H \times H}(G \times G)$. In the case of a type I pair (H, ψ), where the Q-system satisifies chiral locality (or in categorical language the corresponding Frobenius algebra is commutative), the images are isomorphic to each other and to $K^0_{\Delta \times H}(G \times G)$. The neutral system, i.e. the intersection of the images, will not only be commutative, but a non-degenerately braided system – such as $K^0_{\Delta(K) \times \Delta(K)}(K \times K)$ for some subgroup K of G, i.e. isomorphic to the Verlinde algebra of the quantum double of K. For simplification we only discuss this here in the case of the doubles of the finite cyclic groups $\mathbb{Z}_2$, $\mathbb{Z}_3$ and the symmetric group S_3 on three symbols in level zero (i.e. the untwisted case).

2 Twisted Quantum Doubles of Finite Groups

2.1 *G*-kernels and 3-cohomology

Let G be a finite group, and take a G-kernel on an infinite factor M. That is we have a homomorphism from G into the outer automorphism group Out(M) of M, namely the automorphism group Aut(M) of M modulo Int(M) the inner automorphisms of M. If ν_g in Aut(M) is a choice of representatives for each g in G of the G-kernel, then

$$\nu_g \nu_h = \mathrm{Ad}(u(g,h))\nu_{gh} \,,$$

for some unitary $u(g,h)$ in M, for each pair g,h in G. We can assume the normalisation $\nu_e = id_M$, $u(g,e) = u(e,g) = 1_M$, for all g in G, where e is the unit of the group. By associativity of $\nu_g \nu_h \nu_k$, we have a scalar $\omega(g,h,k)$ in $\mathbb{T}$ such that

$$u(g,h)u(gh,k) = w(g,h,k)\nu_g(u(h,k))u(g,hk) \,, \tag{2}$$

i.e. $\omega = \partial_\nu u$, the ν-coboundary of u. A computation [56] shows that ω is a 3-cocycle in $Z^3(G,\mathbb{T})$:

$$\omega(g,h,k)\omega(g,hk,l)\omega(h,k,l) = \omega(gh,k,l)\omega(g,h,kl) \,, \quad g,h,k,l \in G \,. \tag{3}$$

Every element of $Z^3(G,\mathbb{T})$ arises in this way from some G-kernel [56, 40] (see also [42, 60]).

If α, β are two endomorphisms between two algebras we let $\mathrm{Hom}(\alpha,\beta)$ denote the intertwiner space $\{x : x\alpha(a) = \beta(a)x, \forall a\}$ in the target algebra. Then $\mathrm{Hom}(\nu_{\mathrm{gh}}, \nu_{\mathrm{g}}\nu_{\mathrm{h}})$ is a line bundle $P(g,h)$ spanned by $u(g,h)$.

The conjugate $\overline{\nu}_g$ of the automorphism ν_g can be taken to be $\nu_{g^{-1}}$. We define isometries:

$$r_g = u(g^{-1},g), \qquad \overline{r}_g = \overline{\omega}(g,g^{-1},g)u(g,g^{-1}) \,. \tag{4}$$

Then $r_g \in \mathrm{Hom}(1, \overline{\nu_g}\nu_g)$ and $\overline{r}_g \in \mathrm{Hom}(1, \nu_g\overline{\nu_g})$ such that

$$\overline{r}_g^* \nu_g(r_g) = 1_M \,, \qquad r_g^* \overline{\nu_g}(\overline{r}_g) = 1_M \,. \tag{5}$$

Intertwiners can be written graphically in the notation and conventions of [11]. The set $\{\nu_g : g \in G\}$ of automorphisms forms a system of endomorphisms in the sense of [11], and we can form the quantum double system and consequently the associated topological quantum field theory. The vanishing of the 3-cohomology class of ω in $H^3(G,\mathbb{T})$ is precisely when we can adjoin unitaries $\{v_g : g \in G\}$ so that [56]:

$$v_g v_h = u(g,h)v_{gh}, \quad \nu_g(m) = Ad(v_g)(m), g,h \in G, m \in M.$$

In this special case we can form the twisted cross product $M \rtimes G = M \rtimes_\nu G$, and then perform the iterated Jones construction

$$M \subset M \rtimes G \subset M_1 \subset M_2 \subset \ldots$$

and complete to obtain M_∞. The quantum double system in this case is the M_∞-M_∞ system for the subfactor $A = M \vee M' \subset B = M_\infty$.

2.2 Rectangular Algebra

Ocneanu has introduced tube algebras and double triangle algebras for understanding and handling the combinatorics of intertwiner spaces (see e.g. [50, 51, 23, 11]). We need variants of this - a rectangular algebra and a super-tube algebra. First consider the rectangular algebra $\mathcal{R} = \mathcal{R}^\omega(G)$:

$$\mathcal{R}^\omega(G) = \bigoplus_{a,h,k} \mathrm{Hom}(\nu_h\nu_a, \nu_{hak}\overline{\nu}_k)\,. \tag{6}$$

Note that $\mathrm{Hom}(\nu_h\nu_a, \nu_b\overline{\nu}_k)$ vanishes unless $b = hak$, when $\mathrm{Hom}(\nu_h\nu_a, \nu_{hak}\overline{\nu}_k)$ is one dimensional or a line bundle

$$R(h,a,k) \simeq P(hak, k^{-1}) \otimes P(h,a)^*, \tag{7}$$

spanned by the intertwiners

$$r(h,a,k) = u(hak, k^{-1})u(h,a)^*. \tag{8}$$

There is a natural product map

$$\begin{aligned}&\mathrm{Hom}(\nu_{h'}\nu_{a'}, \nu_{h'a'k'}\overline{\nu}_{k'}) \times \mathrm{Hom}(\nu_h\nu_a, \nu_{hak}\overline{\nu}_k)\\ &\quad \to \mathrm{Hom}(\nu_{h'h}\nu_a, \nu_{h'hakk'}\overline{\nu}_{kk'})\,,\end{aligned}$$

given by

$$S' \times S \to \delta_{a',hak}\nu_{h'}(S')S, \tag{9}$$

so that we have the coherence:

$$R(h', hak, k') \otimes R(h,a,k) \simeq R(h'h, a, kk'). \tag{10}$$

In addition, there is an involution

$$\mathrm{Hom}(\nu_\mathrm{h}\nu_\mathrm{a}, \nu_\mathrm{hak}\overline{\nu}_\mathrm{k}) \to \mathrm{Hom}(\nu_{hak}\overline{\nu}_k, \nu_h\nu_a,) \to \mathrm{Hom}(\overline{\nu}_h\nu_{hak}, \nu_a\nu_k)$$

$$S \to S^\dagger = r_h^*\overline{\nu}_h[S^*\nu_{hak}(\overline{r}_{k^{-1}})]\,,$$

obtained by first taking the involution in the von Neumann algebra M and then using Frobenius reciprocities [11]. Here to avoid confusion we denote $*$ as the involution in the algebra M, and $\dagger$ as the involution in the rectangular space. Conseqently,

$$R(h,a,k)^\dagger \simeq R(h^{-1}, hak, k^{-1})\,. \tag{11}$$

It is this involution that we will use to get a $*$-structure on our algebras and when there is no likely confusion we will denote this by $*$ as usual.

However, there is another related conjugate linear automorphism. First, note that there are natural identifications of intertwiner spaces

$$\mathrm{Hom}(\nu_h\nu_b, \nu_{hbk}\overline{\nu}_k) \to \mathrm{Hom}(\nu_{hbk}\overline{\nu}_k, \nu_h\nu_b) \to \mathrm{Hom}(\overline{\nu}_k\overline{\nu}_b, \overline{\nu}_{hbk}\nu_h)$$

$$S \to S^\flat = \overline{\nu}_{hak}[\nu_h(\overline{r}_a^*)S^*)]r_{hak}\,,$$

by first taking adjoints and then using Frobenius reciprocity. This is a conjugate linear automorphism so that

$$R(h,a,k)^\flat \simeq R(k^{-1},a^{-1},h^{-1})\,. \tag{12}$$

This endows $\mathcal{R}^\omega(G)$ as a finite dimensional C^*-algebra. In terms of the canonical generators we have the relations:

$$\begin{aligned} r(h',a',k')r(h,a,k) &= \delta_{a',hak}\omega(h',h,a)\overline{\omega}(h',hak,k^{-1}) \\ &\quad \times\omega(h'hakk',k'^{-1},k^{-1})r(h'h,a,kk')\,, && (13)\\ r(h,a,k)^* &= \overline{\omega}(h^{-1},h,a)\omega(h^{-1},hak,k^{-1}) \\ &\quad \times\overline{\omega}(a,k,k^{-1})r(h^{-1},hak,k^{-1})\,, && (14)\\ r(h,a,k)^\flat &= \quad \omega(k^{-1},a,a^{-1})\omega(k^{-1}a^{-1}h^{-1},hak,k^{-1}) \\ &\quad \times\overline{\omega}(k^{-1}a^{-1}h^{-1},h,a)r(k^{-1},a^{-1},h^{-1})\,. && (15) \end{aligned}$$

2.3 Tube Algebra

We will use this rectangular space to construct the tube algebra and the super-tube algebra. First the tube algebra is the space of intertwiners $\mathcal{D} = \mathcal{D}^\omega(G)$:

$$\mathcal{D}^\omega(G) = \bigoplus_{a,h} \mathrm{Hom}(\nu_h\nu_a, \nu_{hah^{-1}}\nu_h)\,. \tag{16}$$

Here $\mathrm{Hom}(\nu_h\nu_a, \nu_{hah^{-1}}\nu_h)$ is one dimensional or a line bundle

$$C(a,h) \simeq P(hah^{-1},h) \otimes P(h,a)^*, \tag{17}$$

spanned by the intertwiners

$$c(a,h) = r(h,a,h^{-1}) = u(hah^{-1},h)u(h,a)^*. \tag{18}$$

We have the coherence and involutive properties:

$$C(hah^{-1},h') \otimes C(a,h) \simeq C(a,h'h)\,, \tag{19}$$

$$C(a,h)^\dagger \simeq C(hah^{-1},h^{-1})\,, \tag{20}$$

The tube algebra is then a finite dimensional C^*-algebra with generators $\{c(a,h) : a,h\}$ and relations:

$$\begin{aligned} c(a',h')c(a,h) &= \delta_{a',hah^{-1}}w(h',h,a)\overline{w}(h',hah^{-1},h) \\ &\quad \times \quad w(h'hah^{-1}h'^{-1},h',h)c(a,h'h)\,, && (21)\\ c(a,h)^* &= \overline{w}(h^{-1},h,a)w(h^{-1},hah^{-1},h) \\ &\quad \times \quad \overline{w}(a,h^{-1},h)c(hah^{-1},h^{-1})\,. \qquad . && (22) \end{aligned}$$

The group G has two parent algebras. One is the function algebra $C(G)$ of complex valued functions on G under pointwise multiplication $fg(a) = f(a)g(a)$ and co-multiplication $\Delta(f)(a,b) = f(ab)$, spanned by the delta functions $\delta_g(h) = \delta_{g,h}$. The other is its dual $C(G)^*$ as a Hopf algebra, the group algebra $\mathbb{C}(G)$ with multiplication $a \otimes b \to ab$ and co-multiplication $a \to a \otimes a$. The function algebra $C(G)$ and group algebra $\mathbb{C}(G)$ embedd into the tube algebra as $\delta_g \to c(g,e)$, and $h \to c(e,h)$, so that indeed the tube algebra is their tensor product $C(G) \otimes \mathbb{C}(G)$ as a vector space, with $c(g,h)$ identified with $\delta_g \otimes h$, but the product on $\mathcal{D}^\omega(G)$ is twisted by the 3-cocycle ω. The representations of the tube algebra $\mathcal{D}^\omega(G)$ is described by G-equivariant vector bundles over G. If ρ is a representation of $\mathcal{D} = \mathcal{D}^\omega(G)$ on V, then since it is in particular a representation of the function algebra $C(G)$, we can write $V_g = \rho(\delta_g)V$ to give a vector bundle over G.

We can read the coherence or the $\mathcal{D}^\omega(G)$-action on V as G-equivariance expressed as

$$C(a,h) \otimes V_a \simeq V_{hah^{-1}}\,, \tag{23}$$

where $C(a,h)$ is the line bundle $\mathrm{Hom}(\nu_h\nu_a, \nu_{hah^{-1}}\nu_h)$, so that we have maps

$$\pi_h^V : C(a,h) \otimes V_a \to V_{hah^{-1}}$$

$$\pi_h^V(\ell \otimes v_a) = \rho(\ell)v_a.$$

By some abuse of notation we find it convenient to write this as

$$h.v_a = \pi_h^V(c \otimes v_a) = \rho(c)v_a,$$

for the particular interwiner $\ell = c(a,h)$ or cross section as in Eq. (18). Thus $h.v_a \in V_{hah^{-1}}$, for $v_a \in V_a$. We can think of this as one vector space V_a sitting over one end of the tube and $V_{hah^{-1}}$ over the other, with one transported to the other via the line bundle $C(a,h)$. Note that the coherence Eq. (19) of line bundles is reflected as a twisted left action:

$$h'.(h.v_a) = \omega(h',h,a)\overline{\omega}(h',hah^{-1},h)\omega(h'hah^{-1}h'^{-1},h',h)h'h.v_a\,. \tag{24}$$

Elements of Rep($\mathcal{D}^\omega$(G)), the representations of $\mathcal{D}^\omega(G)$, are thus described as vector bundles over G, with a twisted left action satisfying Eq. (24).

To take a fusion product of two representations of $\mathcal{D} = \mathcal{D}^\omega(G)$, we need the co-multiplication operator Δ from $\mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$, which is obtained as follows:

$$\begin{aligned}\Delta c(a,h) = \sum_{a',a'':a'a''=a} & \omega(h,a',a'')\omega(ha'h^{-1},ha''h^{-1},h) \\ & \times\ \overline{\omega}(ha'h^{-1},h,a'')c(a',h) \otimes c(a'',h)\,. \end{aligned} \tag{25}$$

Suppose ρ_{V^1} and ρ_{V^2} are representations of $\mathcal{D}^\omega(G)$ on V^1 and V^2 respectively, then we can define the fusion product representation $\rho_{V^1} \boxtimes \rho_{V^2}$ on $V^1 \otimes V^2$ by:

$$(\rho_{V^1} \boxtimes \rho_{V^2})(x) = (\rho_{V^1} \otimes \rho_{V^2})\Delta(x), x \in \mathcal{D}^\omega(G)\,.$$

This can be interpreted as a product on G-equivariant vector bundles, using

$$\Delta c(a,e) = \sum_{a',a'':a'a''=a} c(a',e) \otimes c(a'',e)\,.$$

We form the vector bundle $V^1 \boxtimes V^2$ by

$$\begin{aligned}(V^1 \boxtimes V^2)_a &= (\rho_{V^1} \boxtimes \rho_{V^2})(c(a,e))(V^1 \boxtimes V^2)\\ &= (\rho_{V^1} \otimes \rho_{V^2})\Delta(c(a,e))(V^1 \otimes V^2) = \oplus_{a'a''=a} V^1_{a'} \otimes V^2_{a''}\,.\end{aligned}$$

The G-action on $V^1 \boxtimes V^2$ is then expressed as

$$\begin{aligned}h.(v^1_{a'} \otimes v^2_{a''}) &= \omega(h,a',a'')\omega(ha'h^{-1},ha''h^{-1},h)\\ &\times \ \overline{\omega}(ha'h^{-1},h,a'')h.v^1_{a'} \otimes h.v^2_{a''}\,,\end{aligned} \tag{26}$$

for $v^1 \in V^1, v^2 \in V^2$.

The trivial bundle $V^0_a = \delta_{a,e}\mathbb{C}$, with trivial action, defines a representation or equivariant bundle V^0 so that $V^0 \boxtimes V^1 \simeq V^1 \simeq V^1 \boxtimes V^0$ for any other bundle V^1.

The tube algebra $\mathcal{D}^\omega(G)$ is a (quasi-associative) Hopf algebra with R-matrix:

$$R = \sum_{a',a''} c(a',e) \otimes c(a'',a')\,.$$

Then the braiding operator is the isomorphism

$$\varepsilon(V^1,V^2) = \tau(\rho_{V^1} \otimes \rho_{V^2})(R) : V^1 \boxtimes V^2 \to V^2 \boxtimes V^1,$$

where τ is the transposition from $V^1 \otimes V^2$ to $V^2 \otimes V^1$. In terms of vector bundles, this braiding takes the form:

$$\varepsilon(V^1,V^2)[v^1_{a'} \otimes v^2_{a''}] = a'.v^2_{a''} \otimes v^1_{a'} \tag{27}$$

for $v^1 \in V^1, v^2 \in V^2$, and is a G-equivariant bundle isomorphism.

This is one picture of the twisted quantum double of Drinfeld of the finite group [1, 2, 18, 37, 49]. It is more convenient for us look at this from another perspective starting in the next section.

2.4 Super-Tube Algebra and Δ-Δ Equivariant Bundles

We want to switch from G-equivariant vector bundles on G to equivariant vector bundles over $G \times G$. For this we consider the super-tube algebra $\mathcal{E} = \mathcal{E}^\omega(G)$, defined as the following finite dimensional C^*-algebra. First note that there are natural identifications of intertwiner spaces

$$\mathrm{Hom}(\nu_h\nu_b, \nu_{hbk}\overline{\nu}_k) \to \mathrm{Hom}(\nu_{hbk}\overline{\nu}_k, \nu_h\nu_b) \to \mathrm{Hom}(\overline{\nu}_k\overline{\nu}_b, \overline{\nu}_{hbk}\nu_h)$$

by first taking adjoints and then using Frobenius reciprocity [11]. Let $S \to S^\flat$ denote the composition of these identifications. We then combine to form the map

$$\mathrm{Hom}(\nu_h\nu_a, \nu_{hak}\overline{\nu}_k) \times \mathrm{Hom}(\nu_h\nu_b, \nu_{hbk}\overline{\nu}_k) \to \mathrm{Hom}(\nu_h\nu_a\overline{\nu}_b, \nu_{hak}\overline{\nu}_{hbk}\nu_h)$$

where

$$(T, S) \to \nu_{hak}(S^\flat)T\,.$$

We denote this intertwiner by $T \times \overline{S}$. Furthermore there is an involution which takes the intertwiner $T \times \overline{S}$ to $T^\dagger \times \overline{S^\dagger}$.

The interwiner space

$$T(h,(a,b),k) = \mathrm{Hom}(\nu_h\nu_a\overline{\nu}_b, \nu_{hak}\overline{\nu}_{hbk}\nu_h) \simeq R(h,a,k) \otimes R(h,b,k)^*$$

has generators

$$\begin{aligned} t(h,(a,b),k) &= \nu_{hak}(r(h,b,k)^\flat)r(h,a,k) \\ &= [r(h,a,k)] \times [r(h,b,k)]^-\,. \end{aligned}$$

We have the coherence and involutive properties:

$$T(h', hak, k') \otimes T(h', a, k) \simeq T(h'h, a, kk')\,, \tag{28}$$

$$T(h,a,k)^\dagger \simeq T(k^{-1}, hak, h^{-1})\,. \tag{29}$$

Consequently, the super-tube algebra $\mathcal{E} = \mathcal{E}^\omega(G)$ defined as

$$\mathcal{E} = \mathcal{E}^\omega(G) = \bigoplus_{a,b,h,k} \mathrm{Hom}(\nu_h\nu_a\overline{\nu}_b, \nu_{hak}\overline{\nu}_{hbk}\nu_h)\,, \tag{30}$$

is a finite dimensional C^*-algebra, with generators $\{t(h,(a,b),k) : h,a,b,k \in G\}$ and relations :

$$\begin{aligned} t(h',x',k')t(h,x,k) &= \delta_{x',hxk}\alpha(h',h,x)\overline{\alpha}(h',hxk,k^{-1}) \\ &\times \quad \alpha(h'hxkk',k'^{-1},k^{-1})t(h'hxkk',k'^{-1},k^{-1})\,, \end{aligned} \tag{31}$$

$$\begin{aligned} t(h,x,k)^\dagger &= \overline{\alpha}(h^{-1},h,x)\alpha(h^{-1},hxk,k^{-1}) \\ &\times \quad \overline{\alpha}(x,k,k^{-1})t(h^{-1},hxk,k^{-1})\,. \end{aligned} \tag{32}$$

for x, y in $\Gamma = G \times G$ and h, k in $\Delta(G)$, the diagonal subgroup identified with G. Here if ω is a 3-cocycle in $Z^3(G,\mathbb{T})$, we define the 3-cocycle $\alpha = \pi_1^*\omega - \pi_2^*\omega$ on Γ if π_1, π_2 are the projections of $G \times G$ on the first and second factors respectively.

We now consider Rep($\mathcal{E}^\omega$(G)), the representations of $\mathcal{E}^\omega(G)$. We claim that these are described by Δ-Δ equivariant twisted vector bundles over $G \times G$. The function algebra on $G \times G$ embedds in $\mathcal{E}^\omega(G)$ by $\delta_x \to t(e,x,e)\,, \quad x \in G \times G$.

Thus if ρ is a representation of $\mathcal{E}^\omega(G)$ on W, it is in particular a representation of the function algebra and so we can write

$$W_x = \rho(\delta_x)W\,, \quad x \in G \times G\,, \tag{33}$$

to get a vector bundle over $G \times G$. The Δ-Δ equivariance is expressed as

$$T(h,x,k) \otimes W_x \simeq W_{hxk}\,, \quad x \in G \times G, \quad h,k \in \Delta(G)\,.$$

In other words, we can act with the diagonal subgroup $G = \Delta(G)$ on the left and right as

$$h.v_x = t(h,x,e)v_x\,, \tag{34}$$

$$v_x.k = t(e,xk,k^{-1})^* v_x\,. \tag{35}$$

So for $v_x \in W_x$, we have $h.v_x \in W_{hx}$, $v_x.k \in W_{xk}$. Then we have the projective relations:

$$h'.(h.v_x) = \alpha(h',h,x)h'h.v_x\,, \tag{36}$$

$$(v_x.k).k' = \overline{\alpha}(x,k,k')v_x.kk'\,. \tag{37}$$

Note that the left and right actions do not in general commute but

$$h.(v_x.k) = \alpha(h,x,k)(h.v_x).k\,, \tag{38}$$

as a consequence of the super-tube relation:

$$t(h,xk,e)t(e,xk,k^{-1})^* = \alpha(h,x,k)t(e,hxk,k^{-1})^* t(h,x,e)\,.$$

To take the fusion product of two representations of Rep($\mathcal{E}^\omega$(G)), we need the co-multiplication operator Δ from $\mathcal{E} \to \mathcal{E} \otimes \mathcal{E}$. The co-multiplication is given by:

$$\begin{aligned}\Delta t(h,x,k) = \sum_{l,x'x''=x} &\alpha(h,x',x'')\alpha(hx'l,l^{-1}x''k,k^{-1}) \\ \times \quad &\overline{\alpha}(hx'l,l^{-1},x'')t(h,x',l) \otimes t(l^{-1},x'',k)\,.\end{aligned} \tag{39}$$

Suppose ρ_{W^1} and ρ_{W^2} are representations of $\mathcal{E}^\omega(G)$ on W^1 and W^2 respectively, then we can define the fusion product representation $\rho_{W^1} \boxtimes \rho_{W^2}$ on $W^1 \otimes W^2$ by:

$$(\rho_{W^1} \boxtimes \rho_{W^2})(x) = (\rho_{W^1} \otimes \rho_{W^1})\Delta(x), x \in \mathcal{E}^\omega(G)\,.$$

This can be interpreted as a product on Δ-Δ-equivariant vector bundles on $G \times G$, using

$$\Delta t(e,x,e) = \sum_{l,x'x''=x} \overline{\alpha}(hx'l,l^{-1},x'')t(e,x',l) \otimes t(l^{-1},x'',e)\,.$$

The bundle is given by

$$(W^1 \boxtimes W^2)_x = (\rho_{W^1} \otimes \rho_{W^2})\Delta(t(e,x,e))(W^1 \otimes W^2) = \oplus_{x'x''=x} W^1_{x'} \otimes W^2_{x''} \,,$$

where we must identify according to the equivalence relation:

$$v_{x'}.l \otimes w_{x''} = \alpha(x',l,x'')v_{x'} \otimes l.w_{x''} \,. \tag{40}$$

Then the Δ-Δ action on $W^1 \boxtimes W^2$ is given by

$$\Delta t(h,x,e) = \sum_{l,x'x''=x} \alpha(h,x',x'')\overline{\alpha}(hx'l,l^{-1},x'')t(h,x',l) \otimes t(l^{-1},x'',e) \,, \tag{41}$$

$$\Delta t(e,x,k) = \sum_{l,x'x''=x} \alpha(x'l,l^{-1}x''k,k^{-1})\overline{\alpha}(x'l,l^{-1},x'') \times \; t(e,x',l) \otimes t(l^{-1},x'',k) \,. \tag{42}$$

Thus the diagonal group Δ acts on the left and right of $W^1 \boxtimes W^2$ by

$$h.(v^1_{x'} \otimes v^2_{x''}) = \alpha(h,x',x'')h.v^1_{x'} \otimes v^2_{x''} \,, \tag{43}$$

$$(v^1_{x'} \otimes v^2_{x''}).k = \overline{\alpha}(x',x'',k)v^1_{x'} \otimes v^2_{x''}.k \,. \tag{44}$$

The trivial bundle $W^0_x = \delta_{x,e}\mathbb{C}$, with trivial action, defines a representation or equivariant bundle W^0 so that $W^0 \boxtimes W^1 \simeq W^1 \simeq W^1 \boxtimes W^0$ for any other bundle W^1.

2.5 Morita Equivalence of G Equivariant and Δ-Δ Equivariant Bundles

We relate the tube algebra $\mathcal{D}$ and the super-tube algebra $\mathcal{E}$ via a Morita equivalence implemented by an intermediary $\mathcal{D}$-$\mathcal{E}$ bimodule $\mathcal{V}$ which we define as

$$\mathcal{V}^\omega(G) = \bigoplus_{a,b,h} \mathrm{Hom}(\nu_h\nu_a\overline{\nu}_b, \nu_{hab^{-1}h^{-1}}\nu_h) \,. \tag{45}$$

We can embedd elements of the tube algebra $\mathcal{D}$ in the super-tube algebra $\mathcal{E}$ by

$$\mathrm{Hom}(\nu_h\nu_a, \nu_{hah^{-1}}\nu_h) \to \mathrm{Hom}(\nu_h\nu_a\overline{\nu}_e, \nu_{hah^{-1}}\overline{\nu}_e\nu_h)$$

by $S \to S \times c(e,h)^-$, if $S \in \mathrm{Hom}(\nu_h\nu_a, \nu_{hah^{-1}}\nu_h)$, so that we identify $c(a,h)$ and $t(h,(a,e),h)$.

Similarly, we can regard an element of the space $\mathcal{V}$ as an element of the super-tube algebra $\mathcal{E}$ by identifying

$$\mathrm{Hom}(\nu_h\nu_a, \nu_{hab^{-1}h^{-1}}\nu_{hb}) \times \mathrm{Hom}(\nu_{hb}\overline{\nu}_b, \nu_h) = \mathrm{Hom}(\nu_h\nu_a\overline{\nu}_b, \nu_{hab^{-1}h^{-1}}\nu_h)$$

in $\mathcal{V}$ with

$$\mathrm{Hom}(\nu_h\nu_a, \nu_{hak}\overline{\nu}_k) \times \mathrm{Hom}(\nu_h\nu_b, \nu_{hbk}\overline{\nu}_k) = \mathrm{Hom}(\nu_h\nu_a\overline{\nu}_b, \nu_{hak}\overline{\nu}_{hbk}\nu_h)$$

in $\mathcal{E}$, for $k = (hb)^{-1}$. Then using the product in the super-tube algebra $\mathcal{E}$, we can regard $\mathcal{V}$ as a $\mathcal{D}$-$\mathcal{E}$ bimodule. This space $\mathcal{V}$ has generators

$$v((a,b),h) = t(h,(a,b),(hb)^{-1}), \quad h,a,b \in G, \tag{46}$$

with the $\mathcal{D}$-$\mathcal{E}$ bimodule relations:

$$\begin{aligned} v(x',h')t(h,x,k) &= \delta_{x',hxk}\alpha(h',h,x)\alpha(h'hxk(h'hbk)^{-1}, h'hbk, k^{-1}) \\ &\quad \times\overline{\alpha}(h,hxk,k)v(x,h'h), \qquad (47) \\ c(x',h')v(x,h) &= \delta_{x',hab^{-1}h^{-1}}\alpha(h',h,x)\alpha(h'hx(h'hb)^{-1}, h', hb) \\ &\quad \times\overline{\alpha}(h', hx(hb)^{-1}, hb)v(x,h'h), \qquad (48) \end{aligned}$$

for $x = (a,b)$.

Thus if W is an $\mathcal{E}$ module, we can form $\mathcal{V}\otimes_{\mathcal{E}} W$ as a natural $\mathcal{D}$ module, and if V is an $\mathcal{D}$ module, we can form $\mathcal{V}^*\otimes_{\mathcal{D}} V$ as a natural $\mathcal{E}$ module. This gives an equivalence of $\mathcal{D}$ and $\mathcal{E}$-modules or a correspondence between G-equivariant vector bundles on G and Δ-Δ equivariant vector bundles on $G \times G$. Let us look at this in more detail at the level of vector bundles. If V is a $\mathcal{D}$-module or a G-equivariant twisted vector bundle, then we form a vector bundle over $G \times G$ by:

$$W_{a,b} = P(a,b^{-1}) \otimes V_{ab^{-1}}, \tag{49}$$

where P is the vector bundle $P(a,b^{-1}) = \mathrm{Hom}(\nu_{\mathrm{ab}^{-1}}, \nu_{\mathrm{a}}\overline{\nu_{\mathrm{b}}})$, as before. Then W becomes an $\mathcal{E}$-module by the coherent actions:

$$\begin{aligned} T(h,(a,b),k) \otimes W_{a,b} &\simeq T(h,(a,b),k) \otimes P(a,b^{-1}) \otimes V_{ab^{-1}} \\ &\simeq P(hak,(hbk)^{-1}) \otimes C(ab^{-1},h) \otimes V_{ab^{-1}} \\ &\simeq P(hak,(hbk)^{-1}) \otimes V_{hab^{-1}h^{-1}} \simeq W_{hak^{-1},hbk^{-1}}. \end{aligned}$$

In particular if $v \in V$, we define $w \in W$ by

$$w_{a,b} = v((a,b),e)^* \otimes v_{ab^{-1}}. \tag{50}$$

Then W becomes a Δ-Δ equivariant twisted bundle by the actions:

$$h.w_{a,b} = \alpha(h^{-1},h,x)\overline{\alpha}(xb^{-1},h^{-1},hb)\alpha(h^{-1},hxb^{-1},hb) \tag{51}$$

$$\times v(h(a,b),e) \otimes h.v_{ab^{-1}}, \tag{52}$$

$$w_{a,b}.k = \overline{\omega}(ab^{-1},b,k^{-1})v((a,b)k,e) \otimes v_{ab^{-1}}. \tag{53}$$

Conversely, suppose that W is an $\mathcal{E}$-module or an equivariant Δ-Δ twisted bundle over $G \times G$, then we can form a bundle V over G by:

$$V_d = \oplus_{ab^{-1}=d} P(a,b^{-1})^* \otimes W_{a,b}. \tag{54}$$

This becomes a $\mathcal{D}$ module by the coherence:

$$C(d,h) \otimes V_d \simeq \oplus_{ab^{-1}=d} C(d,h) \otimes P(a,b^{-1})^* \otimes W_{a,b}$$
$$\simeq \oplus_{ab^{-1}=d} P(ha,(hb)^{-1})^* \otimes W_{ha,hb} \simeq V_{hdh^{-1}}\,.$$

In particular if $w \in W$, we define $v \in V$ by

$$v_d = \oplus_{ab^{-1}=d} v((a,b),e) \otimes w_{a,b}\,. \tag{55}$$

Then V becomes a G equivariant twisted vector bundle over G by

$$h.v_d = \oplus_{ab^{-1}=d} \overline{\omega}(h,ab^{-1},b)\omega(hab^{-1}h^{-1},h,b)v(h(a,b),e) \otimes h.w_{a,b}\,. \tag{56}$$

This gives a (Morita) equivalence of $Rep(\mathcal{D}^\omega(G))$ and $Rep(\mathcal{E}^\omega(G))$, because $\mathcal{V} \otimes_{\mathcal{E}} \mathcal{V}^* = \mathcal{D}$ and $\mathcal{V}^* \otimes_{\mathcal{D}} \mathcal{V} = \mathcal{E}$, cf [54].

The multiplicative properties of these equivalences are as follows. Suppose V^1, V^2 are $\mathcal{D}$ modules. Then we form the fusion product module $V = V^1 \boxtimes V^2$, and the corresponding $\mathcal{E}$ modules, $W^1 = \mathcal{V}^* \otimes_{\mathcal{D}} V^1, W^2 = \mathcal{V}^* \otimes_{\mathcal{D}} V^1$ and $W = \mathcal{V}^* \otimes_{\mathcal{D}} V$ respectively. Then

$$W_{a,b} \simeq P(a,b^{-1}) \otimes V_{ab^{-1}} \simeq P(a,b^{-1}) \otimes (V^1 \boxtimes V^2)_{ab^{-1}}$$
$$\simeq \oplus_{ab^{-1}=xy^{-1}} P(a,b^{-1}) \otimes P(x,y^{-1})^* \otimes V^1_x \otimes V^2_{y^{-1}}$$
$$\simeq \oplus_{a=a'a''} P(a',a'')^* \otimes P(a'',b^{-1}) \otimes V^1_{a'} \otimes V^2_{a''b^{-1}}$$
$$\simeq \oplus_{a=a'a''} P(a',a'')^* \otimes W^1_{a',e} \otimes W^2_{a'',b} \simeq (W^1 \boxtimes W^2)_{a,b}\,,$$

so that $W \simeq W^1 \boxtimes W^2$.

On the other hand suppose that W^1, W^2 are $\mathcal{E}$ modules. Then we form the fusion product module $W = W^1 \boxtimes W^2$, and the corresponding $\mathcal{D}$ modules, $V^1 = \mathcal{V} \otimes_{\mathcal{E}} W^1, V^2 = \mathcal{V} \otimes_{\mathcal{E}} W^2$ and $V = \mathcal{V} \otimes_{\mathcal{E}} W$ resepectively. Then

$$V_d \simeq \oplus_{ab^{-1}=d} P(a,b^{-1})^* \otimes W_{a,b} \simeq \oplus_{ab^{-1}=d} P(a,b^{-1})^* \otimes (W^1 \boxtimes W^2)_{a,b}$$
$$\simeq \oplus_{ab^{-1}=d} \oplus_{a'a''=a} P(a,b^{-1})^* \otimes P(a',a'')^* \otimes W^1_{a',e} \otimes W^2_{a'',b}$$
$$\simeq \oplus_{d'd''=d} \oplus_{a''b^{-1}=d''} P(d',d'')^* \otimes P(a'',b^{-1})^* \otimes W^1_{d',e} \otimes W^2_{a'',b}$$
$$\simeq \oplus_{d'd''=d} P(d',d'')^* \otimes V^1_{d'} \otimes V^2_{d''} \simeq (V^1 \boxtimes V^2)_d\,,$$

so that $V \simeq V^1 \boxtimes V^2$.

2.6 Braiding on Δ-Δ Equivariant Bundles

We can use this Morita equivalence to understand the braiding on the representations of super-tube algebra $\mathcal{E}$. Thus the Δ-Δ twisted equivariant bundles on $G \times G$ becomes a braided tensor category.

Suppose that W^1, W^2 are $\mathcal{E}$ modules. Then we form the fusion product module $W = W^1 \boxtimes W^2$, and the corresponding $\mathcal{D}$ modules, $V^1 = \mathcal{V} \otimes_{\mathcal{E}} W^1, V^2 = \mathcal{V} \otimes_{\mathcal{E}} W^2$ and $V = \mathcal{V} \otimes_{\mathcal{E}} W$ resepectively, so that $W^1 \boxtimes W^2 \simeq$

$\mathcal{V} \otimes_{\mathcal{D}} V^1 \boxtimes V^2$. The braiding on $\mathrm{Rep}(\mathcal{E}^\omega(\mathrm{G}))$ is then simply: $\varepsilon(W^1, W^2)$ from $W^1 \boxtimes W^2 \to W^2 \boxtimes W^1$ given by

$$\varepsilon(W^1, W^2) = 1_{\mathcal{V}} \otimes_{\mathcal{D}} \varepsilon(V^1, V^2) : \mathcal{V} \otimes_{\mathcal{D}} V^1 \boxtimes V^2 \to \mathcal{V} \otimes_{\mathcal{D}} V^2 \boxtimes V^1$$

To be more explicit in terms of twisted vector bundles, we first have:

$$\begin{aligned}(V^1 \boxtimes V^2)_d &\simeq \oplus_{d'd''=d} P(d', d'')^* \otimes V^1_{d'} \otimes V^2_{d''} \\ &\simeq \oplus_{d'd''=d} \oplus_{a''b^{-1}=d''} P(d', d'')^* \otimes P(a'', b^{-1})^* \otimes W^1_{d',e} \otimes W^2_{a'',b}\,,\end{aligned}$$

If $v^1 \in V^1, v^2 \in V^2$, then according to Eq. (27), we have

$$\varepsilon(V^1, V^2)[v^1_{d'} \otimes v^2_{d''}] = d'.v^2_{d''} \otimes v^1_{d'} \quad \in V^2_{d'd''d'^{-1}} \otimes V^1_{d'} \tag{57}$$

If $w^1 \in W^1, w^2 \in W^2$, we define $v^1 \in V^1, v^2 \in V^2$ by

$$v^1_{d'} = v((d', e), e) \otimes w^1_{d',e}\,, \tag{58}$$

$$v^2_{d''} = v((d'', e), e) \otimes w^2_{d'',e}\,. \tag{59}$$

Writing $v = v^2$, $w = w^2$, we compute:

$$\begin{aligned}d'v_{d''} &= c(d'', d')v((d'', e), e) \otimes w_{d'',e} \\ &= v((d'd''d'^{-1}, e), e)t(d', (d'', e), d')w_{d'',e} \\ &= v((d'd''d'^{-1}, e), e)t(e, d'(d'', e), d')t(d', (d'', e), e)w_{d'',e} \\ &= v((d'd''d'^{-1}, e), e)\omega(d'd'', d'^{-1}, d')\overline{\omega}(d', d'^{-1}, d')(d'w^2_{d'',e})d'^{-1}\,.\end{aligned}$$

Untangling this, we get that

$$\begin{aligned}&\varepsilon(W^1, W^2)[w^1_{d',e} \otimes w^2_{d'',e}] = \\ &\omega(d'd'', d'^{-1}, d')\overline{\omega}(d', d'^{-1}, d')(d'w^2_{d''})d'^{-1} \otimes w^1_{d',e} \in W^2_{d'd''d'^{-1},e} \otimes W^1_{d',e}\,.\end{aligned} \tag{60}$$

By equivariance, this is enough to compute the brading on anything. For example, we have:

$$\begin{aligned}&\varepsilon(W^1, W^2)[w^1_{e,b} \otimes w^2_{a,e}] \\ &= \omega(b, b^{-1}, b)\varepsilon(W^1, W^2)[b(b^{-1}w^1_{e,b}) \otimes w^2_{a,e}] \\ &= \omega(b, b^{-1}, b)\overline{\omega}(b, b^{-1}, a)\varepsilon(W^1, W^2)[b(b^{-1}w^1_{e,b} \otimes w^2_{a,e})] \\ &= \omega(b, b^{-1}, b)\overline{\omega}(b, b^{-1}, a)b\varepsilon(W^1, W^2)[b^{-1}w^1_{e,b} \otimes w^2_{a,e}] \\ &= \omega(b, b^{-1}, b)\overline{\omega}(b, b^{-1}, a)\omega(b^{-1}a, b, b^{-1})\overline{\omega}(b^{-1}, b, b^{-1})b.[(b^{-1}w^2_{a,e})b \otimes b^{-1}w^1_{e,b}] \\ &= \omega(b, b^{-1}, b)\overline{\omega}(b, b^{-1}, a)\overline{\omega}(b^{-1}, b, b^{-1})b.[b^{-1}w^2_{a,e} \otimes b(b^{-1}w^1_{e,b})] \\ &= \overline{\omega}(b, b^{-1}, a)\overline{\omega}(b^{-1}, b, b^{-1})b.[b^{-1}w^2_{a,e} \otimes w^1_{e,b}] \\ &= \overline{\omega}(b, b^{-1}, a)[b.(b^{-1}w^2_{a,e}) \otimes w^1_{e,b}] \\ &= w^2_{a,e} \otimes w^1_{e,b}\,.\end{aligned}$$

We record this as:

$$\varepsilon(W^1, W^2)[w^1_{e,b} \otimes w^2_{a,e}] = w^2_{a,e} \otimes w^1_{e,b}\,. \tag{61}$$

In the case of trivial twisting $\omega = 1$, we have

$$\varepsilon(W^1, W^2)[w^1_{a_1,b_1} \otimes w^2_{a_2,b_2}] = a_1 w^2_{a_2,b_2} b_2^{-1} \otimes a_1^{-1} w^1_{a_1,b_1} b_2\,. \tag{62}$$

Taking inverses, $\varepsilon(W^1, W^2)^{-1} : W^2 \boxtimes W^1 \to W^1 \boxtimes W^2$ will then be determined as:

$$\varepsilon(W^1, W^2)^{-1}[w^2_{a_2,b_2} \otimes w^1_{a_1,b_1}] = b_2 w^1_{a_1,b_1} a_1^{-1} \otimes b_2^{-1} w^2_{a_2,b_2} a_1\,. \tag{63}$$

3 Twisted Equivariant Bundles over Finite Groups

Let $\varGamma$ be a finite group, α a 3-cocycle in $Z^3(\varGamma, \mathbb{T})$, with H and K subgroups of $\varGamma$, and ψ and ψ' are 2-cocycles in $Z^2(H, \mathbb{T})$, $Z^2(K, \mathbb{T})$ respectively. We consider a H-K bundle V twisted by α with base space $\varGamma$, where H and K act on the fibres on the left and right with multipliers ψ and ψ' respectively, satisfying the following consistency relations:

$$(h_1 h_2)w = \overline{\alpha}(h_1, h_2, g)\psi(h_1, h_2)(h_1(h_2 w))\,, \tag{64}$$

$$w(k_1 k_2) = \alpha(g, k_1, k_2)^{-1}\psi'(k_1, k_2)((w k_1)(k_2)\,, \tag{65}$$

$$h(wk) = \alpha(h, g, k)(hw)k\,, \tag{66}$$

where $h_1, h_2, h \in H$, $k_1, k_2, k \in K$, and $w = w_g \in V_g$, the fibre over $g \in \varGamma$. Here hw, wk lie in the fibres over hg, and gk respectively, etc. We let ${}^\alpha Bun^{H-K}(\varGamma)$ denote such twisted bundles.

The equivalence classes generate the equivariant twisted K-group ${}^\alpha K^0_{H\times K}(\varGamma)$. If $V$is an H-K bundle, we can naturally associate the conjugate K-H bundle V^*. If L is another subgroup of $\varGamma$, we can naturally form from an H-K bundle V and a K-L bundle W a H-L bundle $V \otimes_K W$:

$${}^\alpha Bun^{H\times K}(\varGamma) \times {}^\alpha Bun^{K\times L}(\varGamma) \to {}^\alpha Bun^{H\times L}(\varGamma) \tag{67}$$

and hence a product on K-theory:

$${}^\alpha K^0_{H\times K}(\varGamma) \times {}^\alpha K^0_{K\times L}(\varGamma) \to {}^\alpha K^0_{H\times L}(\varGamma). \tag{68}$$

We divide the tensor product $V \otimes W$ over $\varGamma \times \varGamma$ by the relation:

$$v_a k \otimes w_b = \alpha(a, k, b) v_a \otimes k w_b \tag{69}$$

and then push forward under the product map $\varGamma \times \varGamma \to \varGamma$ to obtain $V \otimes_K W$, a bundle over $\varGamma$ where $(V \otimes_K W)_g = \oplus_{ab=g} V_a \otimes W_b$. Then $V \otimes_K W$ becomes a H-K α twisted bundle ${}_H V \otimes_K W_L$ under the natural actions:

$$h(v_a \otimes w_b) = \alpha(h, a, b) h v_a \otimes w_b \tag{70}$$

$$(v_a \otimes w_b)l = \overline{\alpha}(a, b, l) v_a \otimes w_b l \tag{71}$$

Now if V is a bundle, we let $s(V)$ denote its support $\{g \in \Gamma : V_g \neq 0\}$. For an irreducible bundle, the support $s(V)$ is a single double coset HgK. To compute the equivariant K-group ${}^{\alpha}K^0_{H\times K}(\Gamma)$, we first take representatives for the double cosets $H \setminus \Gamma / K$. Then for each double coset HgK we consider the stabilizer subgroup $H \times_g K = \{(h,k) \in H \times K : hg = gk\}$ which is isomorphic to $H \cap {}^gK$ and $H^g \cap K$, where ${}^gK = gKg^{-1}, H^g = g^{-1}Hg$, under the projections $(h, g^{-1}hg) = (gkg^{-1}, k) \to$ to h and k respectively as h determines k and vice versa. Then $(h,k) : w_g \to h(w_g k)$ gives a projective representation of $H \times_g K$ on V_g, with multiplier or 2-cocycle

$$\begin{aligned}\alpha^g(h,h') := {} & \psi_1(h,h')\psi_2(g^{-1}{h'}^{-1}g, g^{-1}h^{-1}g)\overline{\alpha}(hh'g, g^{-1}{h'}^{-1}g, g^{-1}h^{-1}g) \\ & \times\alpha(h,h',g)\alpha(h,h'g,g^{-1}{h'}^{-1}g)\,.\end{aligned}$$

Then the irreducible bundles are labelled by a coset and an irreducible projective representation of the stabilizer.

Suppose G is a finite group and let $\Gamma = G \times G$, and $\Delta(G) = \{(g,g) : g \in G\}$ denote the diagonal subgroup which we denote simply by Δ when there is no confusion. If ω is a 3-cocycle in $Z^3(G,\mathbb{T})$, we define the 3-cocycle $\alpha = \pi_1^*\omega - \pi_2^*\omega$ on $G \times G$ if π_1, π_2 are the projections of $G \times G$ on the first and second factors respectively.

The Verlinde algebra ${}_N\mathcal{X}_N$ for the quantum double of G is then the space of Δ-Δ bundles or the equivariant K-group $K^0_{\Delta\times\Delta}(G \times G)$. Since

$$\Delta(G)(g,h)\Delta(G) = \Delta(G)(gh^{-1},1)\Delta(G)$$

for any $g, h \in G$, it is the case that every double coset $\Delta(G)(g,h)\Delta(G)$ gives rise to a conjugacy class $C_{gh^{-1}}$ of $\Delta(G)$. Moreover, the stabilizer of (g,h) equals the centraliser of gh^{-1}, i.e.

$$(g,h)\Delta(G)(g^{-1},h^{-1}) \cap \Delta(G) = \{x \in G : xgh^{-1} = gh^{-1}x\}.$$

Consequently, the primary fields or irreducible bundles are given by pairs (a,χ) where a are representatives of conjugacy classes of G and χ are irreducible representations of the centraliser $C_G(a)$ of $a \in G$.

There are a number of special cases of particular interest. One is when $H = \Delta$, and $\psi = 1$. The double cosets $\Delta \setminus \Gamma / K$ are labelled by

$$\Delta(G)(g,h)K = \Delta(G)(gh^{-1},1)K$$

i.e. of the form $\Delta(x,1)K$ where x in G is defined up to an action of $(h,k) \in K \times K$ by conjugation $x \to hxk^{-1}$. For each such x, we identify the stabiliser subgroup $\Delta \times_{(x,1)} K$ with the subgroup $K^x = \{(h_1,h_2) \in K : h_1xh_2^{-1} = x\}$ of K. Again, since h_2 is determined by h_1 the group K^x can be understood as a subgroup of G through projecting $K \subset G \times G$ to the first component. The multiplier $\omega^{(x,1)}$ of the subgroup $\Delta \times_{(x,1)} K$ is then regarded as a multiplier on K^x.

3.1 The Frobenius Algebra

For the remaining exposition, we take for simplicity the case of trivial twist or level zero $\omega = 1$ in $Z^3(G, \mathbb{T})$, for a finite group G. The quantum double of the group G is identified with the inclusion $N = M_0 \rtimes \Delta(G) \subset M_0 \rtimes (G \times G) = M = M_\Delta$ where $\Delta(G) = \{(g,g) : g \in G\}$ denotes the diagonal subgroup of $G \times G$. The N-N system is described by $Bun^{\Delta-\Delta}(\Gamma)$, if $\Gamma = G \times G$ [45, 46, 25].

For H a subgroup of $G \times G$, we define an irreducible element ι, a bundle in $Bun^{H-\Delta}(\Gamma)$, using the trivial representation on the trivial double coset:

$$\iota = \iota_H = [H\Delta, 0]$$

and similarly,

$$\bar{\iota} = \bar{\iota}_H = [\Delta H, 0]$$

in $Bun^{\Delta-H}(\Gamma)$. Again for simplicity, we take $\psi = 1$ in $Z^2(H, \mathbb{T})$, We compute the products using [46]:

$$\theta = \bar{\iota}\iota = \sum_{h \in \Delta \cap H \backslash H / \Delta \cap H} [\Delta h \Delta, Ind(0)^{\Delta^h \cap \Delta}_{\Delta^h \cap H \cap \Delta}] \,,$$

$$\gamma = \iota\bar{\iota} = \sum_{k \in \Delta \cap H \backslash \Delta / \Delta \cap H} [HkH, Ind(0)^{H^k \cap H}_{H^k \cap H \cap \Delta}] \,,$$

in $Bun^{\Delta-\Delta}(\Gamma)$, $Bun^{H-H}(\Gamma)$ respectively. The former yields a Frobenius algebra $\Theta = \Theta_H$, or Q-system in the braided tensor category $Bun^{\Delta-\Delta}(\Gamma)$. Since this is identified with the N-N system, we thus have a subfactor $N \subset M_H$. Thus if we have two such subgroups H_a and H_b of Γ, then the corresponding M_a-M_b system is identified with $Bun^{H_a-H_b}(\Gamma)$ (the irreducible components of $\{\iota_a \lambda \bar{\iota}_b : \lambda\}$). In particular, we identify the N-M_H sectors with $Bun^{\Delta-H}(\Gamma)$ and the M_H-M_H system with $Bun^{H-H}(\Gamma)$.

In particular, for the special cases:

$$\Delta \subset H : \bar{\iota}\iota = \sum_{h \in \Delta \backslash H / \Delta} [\Delta h \Delta, 0] = [H, 0] \,,$$

$$H \subset \Delta : \bar{\iota}\iota = [\Delta, Ind(0)^{\Delta}_{H}] \,.$$

3.2 α-Induction and Modular Invariants

We can use the Frobenius algebra $\Theta = \Theta_H$ or Q-system for each H to define α-induction from $Bun^{\Delta-\Delta}(\Gamma)$ to $Bun^{H-H}(\Gamma)$ and hence construct a modular invariant.

We identify $Bun^{H-\Delta}(\Gamma)$, with left Θ modules, $Bun^{\Delta-H}(\Gamma)$ with right Θ modules and $Bun^{H-H}(\Gamma)$, with Θ-Θ bimodules. More generally, if $\Theta_a = \Theta_{H_a}$, and $\Theta_b = \theta_{H_b}$, we identify following [25] $Bun^{H_a-H_b}(\Gamma)$ with Θ_a-Θ_b bimodules.

Recall that a Θ_a-Θ_b bimodule [50, 25] is an element M of $Bun^{\Delta-\Delta}(\Gamma)$ with morphisms from $\theta_a \otimes_G M$ and $M \otimes_G \theta_b$ into M satisfying natural compatibility conditions.

Every irreducible β in $Bun^{H_a-H_b}(\Gamma)$ arises from the decomposition $\iota_a \lambda \bar{\iota}_b$ with λ in $Bun^{\Delta-\Delta}(\Gamma)$. Define now $\Phi : Bun^{H_a-H_b}(\Gamma) \to \Theta_a$-$\Theta_b$-bimodules by $\Phi(\beta) = \bar{\iota}_a \beta \iota_b$ for $\beta \in Bun^{H_a-H_b}(\Gamma)$. In particular, $\Phi(\iota_a \lambda \bar{\iota}_b) = \theta_a \lambda \theta_b$. If $\beta, \beta' \in Bun^{H_a-H_b}(\Gamma)$, then we map an intertwiner $t \in \mathrm{Hom}(\beta, \beta')$ to $\Phi(t) = \bar{\iota}_a t \iota_b \in \mathrm{Hom}(\bar{\iota}_a \beta \iota_b, \bar{\iota}_a \beta' \iota_b) = \mathrm{Hom}(\Phi(\beta), \Phi(\beta'))$. Then Φ is a Θ_a-Θ_b morphism. We note that Φ is injective. Suppose that $\bar{\iota}_a \beta \iota_b \simeq \bar{\iota}_a \beta' \iota_b$ as Θ_a-Θ_b bimodules. Then $t = \mathbf{1}_{\bar{\iota}_a} \otimes t'' \otimes \mathbf{1}_{\iota_b}$ with $t'' \in \mathrm{Hom}(\beta, \beta')$. If t is an isomorphism so is t'', therefore $\beta \simeq \beta'$. That Φ is surjective can be seen by counting dimension (cf. proof of Lemma 3.1 in [25]).

In this formulation, α-induction looks as follows. Take a bundle V in $Bun^{\Delta-\Delta}(\Gamma)$, and form the bundle $V \otimes_G \theta$ again in $Bun^{\Delta-\Delta}(\Gamma)$. The latter determines a Θ-Θ bimodule, since θ itself is a Θ-Θ bimodule. Consequently, $V \otimes_G \theta$ becomes a Θ-Θ bimodule using the natural action of θ on the right on θ, and the braiding $\varepsilon^{\pm}(V, \theta)$ to hit θ in $V \otimes_G \theta$ on the left. We can then identify these induced bimodules $\alpha_V^{\pm}$ with elements of $Bun^{H-H}(\Gamma)$ using the previous paragraph.

In the special case when $\Delta \subset H \subset \Gamma$, we can also view this construction as follows. For a bundle V in $Bun^{\Delta-\Delta}(\Gamma)$ we form the product Δ-Δ bundle $V \otimes_\Delta \theta$. This can be considered as a H-H bundle since θ can be viewed as a H-H bundle, and so $V \otimes_\Delta \theta$ has a natural right H action on θ, and using the braiding to identify $V \otimes_\Delta \theta$ with $\theta \otimes_\Delta V$, which has a natural left H action. Since we can use the braiding $\varepsilon^+ = \varepsilon(V, \theta)$ or its adjoint $\varepsilon^- = \varepsilon(\theta, V)^*$, we can form two inductions $\alpha^{\pm}$ in this way. Note that the relation $\iota \otimes_G V \simeq \alpha_V^{\pm} \otimes_H \iota_G$ as H-G bundles since $\iota \otimes_G V \simeq (V \otimes_G \bar{\iota}\iota) \otimes_H \iota$, as $\theta \otimes_H \iota \simeq \iota$ as H-G bundles.

Any such subgroup $\Delta \subset H \subset \Gamma$, is of the form $H = \Delta(1, N) = \Delta(N, 1)$, where N is a normal subgroup of G (or Δ). Indeed $N = p(H)$, where p is the projection $(g_1, g_2) \to g_1 g_2^{-1}$ from Γ to G. Clearly any such subgroup H is invariant under the flip σ on Γ. We will see in the examples of Sect. 4 that the neutral system can be identified with the non-degenerately braided system, $Bun^{\Delta(N)-\Delta(N)}(N \times N)$, i.e. the quantum double of N.

We compute explicitly the induced bundle $\alpha_V^{\pm}$ when $V = [\Delta a \Delta, \chi]$ is an irreducible Δ-Δ bundle, with $a \in \Gamma$ and a representation π of the stabiliser $\Delta_a = \Delta \cap {}^a\Delta$ with character χ. We have

$$\begin{aligned}
h^+(v_{a_1 b_1} \otimes e_{a_2 b_2}) &= \varepsilon(V,\theta)^{-1}[h(\varepsilon(V,\theta)(a_1(e_{a_2,b_2} b_2^{-1}) \otimes (a_1^{-1} v_{a_1,b_1}) b_2)] \\
&= \varepsilon(V,\theta)^{-1}[h.(a_1(e_{a_2,b_2} b_2^{-1}) \otimes (a_1^{-1} v_{a_1,b_1}) b_2)] \\
&= \varepsilon(V,\theta)^{-1}[h a_1(e_{a_2,b_2} b_2^{-1})) \otimes (a_1^{-1} v_{a_1,b_1}) b_2] \\
&= h_1 a_1 [a_1^{-1} v_{a_1 b_1} b_2] b_2^{-1} \otimes (h_1 a_1)^{-1} [h a_1 (e_{a_2,b_2} b_2^{-1})] b_2 \\
&= h_1 v_{a_1 b_1} (ad(a_1^{-1})(h_1) \otimes (ad(a_1^{-1}(h))(e_{a_2,b_2}) \,.
\end{aligned}$$

We arrive at:

$$\alpha^{\pm}_{[\Delta a\Delta,\pi]} = [HaH, \mathrm{Ind}^{\Delta\cap \mathrm{aHa}^{-1}}_{\Delta\cap \mathrm{a}\Delta\mathrm{a}^{-1}}(\pi)\pi^{\pm}] = [\mathrm{HaH}, \pi_{\pm}]\,, \tag{72}$$

for $\pi \in \mathrm{Rep}[\Delta \cap \mathrm{a}\Delta\mathrm{a}^{-1}]$ where $\pi^{\pm} : (h_+, h_-) \to (h_+, h_+)\,\mathrm{or}\,(\mathrm{h}_-, \mathrm{h}_-)$ takes $H \cap aHa^{-1} \to \Delta \cap aHa^{-1}$.

Let us examine this case $\Delta \subset H \subset \Gamma$ when G is abelian. Then

$$\pi_+(h_1, h_2) = \pi(h_1),\ \pi_-(h_1, h_2) = \pi(h_2).$$

This means that:

$$\begin{aligned}\alpha^{+}_{[a\Delta,\chi]} &= [aH, (\chi \times 1)|_H]\,,\\ \alpha^{-}_{[a\Delta,\chi]} &= [aH, (1 \times \chi)|_H]\,.\end{aligned} \tag{73}$$

Now we can write $H = \Delta \times N$, as sets using the identification $(a, b) \to (a, a^{-1}b) = (\delta, n)$, so that $(\chi \times 1)|_H(a, b) = \chi(\delta)$, $(1 \times \psi)|_H(a, b) = \psi(\delta)\psi(n)$. Then

$$\langle(\chi \times 1)|_H, (\psi \times 1)|_H\rangle = \langle\chi, \psi\rangle\,,$$
$$\langle(\chi \times 1)|_H, (1 \times \psi)|_H\rangle = \langle\chi, \psi\rangle\langle\psi|_N, 1\rangle\,.$$

Consequently, $\langle\alpha^{+}_{[a\Delta,\chi]}, \alpha^{+}_{[b\Delta,\psi]}\rangle = 1$, if $ab^{-1} \in H$, $\chi = \psi$, and 0 otherwise. Moreover, the modular invariant mass matrix is given as $\langle\alpha^{+}_{[a\Delta,\chi]}, \alpha^{-}_{[b\Delta,\psi]}\rangle = 1$ if $ab^{-1} \in H$, $\chi = \psi$ and $\chi|_N = 1$, and 0 otherwise.

4 Examples

For a cyclic groups $\mathbb{Z}_d$, the primary fields are parametrized by pairs (m, n) for $m, n \in \mathbb{Z}_d$ (the first factor labels the double cosets and the second the stabilisers) whose conjugate is $(-m, -n)$, and the S and T matrices:

$$\begin{aligned}S_{(m,n),(m',n')} &= d^{-1}\mathrm{exp}[-2\pi\sqrt{-1}(nm' + mn')/d]\,,\\ T_{(m,n),(m,n)} &= \mathrm{exp}[(2\pi\sqrt{-1}nm/d)]\,.\end{aligned} \tag{74}$$

When $d = p$ is a prime number, the complete list of all modular invariants is described in [16, 27]. There are four non-permutation modular invariants: $\mathcal{Z}_4 = xx^*, \mathcal{Z}_7 = xy^*$, $\mathcal{Z}_8 = yx^*$, and $\mathcal{Z}_5 = yy^*$ where $x = \sum_{i=0}^{p-1}\chi_{i0}$ and $y = \sum_{j=0}^{p-1}\chi_{0j}$. For any prime $p > 2$, there will be four permutation modular invariants: $\mathcal{Z}_1 = \sum_{i,j=0}^{p-1}\chi_{ij}\chi^*_{ij}$, $\mathcal{Z}_2 = \sum_{i,j=0}^{p-1}\chi_{ij}\chi^*_{ji}$, $\mathcal{Z}_3 = \sum_{i,j=0}^{p-1}\chi_{ij}\chi^*_{-j,-i}$ and $\mathcal{Z}_6 = \sum_{i,j=0}^{p-1}\chi_{ij}\chi^*_{-i,-j}$ the charge conjugation. When $p = 2$, $\mathcal{Z}_6 = \mathcal{Z}_1$ and $\mathcal{Z}_3 = \mathcal{Z}_2$ and we have six distinct modular invariants. When $p = 3$, there are precisely eight distinct modular invariants. In the cases $p = 2, 3$, these exhaust all the modular invariants.

4.1 $G = \mathbb{Z}_2$

Here there are six distinct modular invariants, four of which are symmetric. All can be realised from subfactors or from module categories. The following are the 5 subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$\begin{aligned} H_1 &= \{(0,0)\}\,, \\ H_3 &= \{(0,0),(1,1)\} = \Delta(\mathbb{Z}_2)\,, \\ H_4 &= \mathbb{Z}_2 \times \{0\}\,, \\ H_5 &= \{0\} \times \mathbb{Z}_2\,, \\ H_6 &= \mathbb{Z}_2 \times \mathbb{Z}_2\,. \end{aligned}$$

In this case the 2-cohomology groups $H^2(H, \mathbb{T})$ are all trivial except for H_6 when it is $\mathbb{Z}_2$ and only H_1, H_3 and H_6 give rise to type I modular invariants or commutative Q-systems.
(i) *Example:* H_3 and H_6.
The subgroups H_3 and H_6 contain Δ with $N = 0$, and $N = G = \mathbb{Z}_2$ respectively. We see that the corresponding modular invariants are $\mathcal{Z} = 1$, and $X_c = xx^* = \mathcal{Z}_4$.
(ii) *Example:* H_1.
The subgroup $H_1 \subset \Delta$. To understand α-induction we first take $V = [x, \psi] \to \iota[x, \psi]\bar{\iota}$. Now $Bun^{H_1 - H_1}(\Gamma) \simeq Bun(\Gamma)$, and $\alpha^{\pm}$-induction takes $[x, \psi] \to [(x, 0)], [(0, x)]$ respectively. Consequently, the corresponding modular invariant is $\mathcal{Z}_5 = yy^* = X_s$.
(iii) *Example:* H_4 and H_5.
There are two further non symmetric groups $H_4 = \mathbb{Z}_2 \times \{0\}$, and $H_5 = \{0\} \times \mathbb{Z}_2$. Writing $H = H_4$, there are two double cosets $H(0,0) = \mathbb{Z}_2 \times \{0\}$ and $H(0,1) = \mathbb{Z}_2 \times \{1\}$, both with stabilisers H, so that there are four irreducible objects in $Bun^{H_4 - H_4}(\Gamma)$ which we write as $[[i, j]]$, where $i = 0, 1$ represents the double coset $H(0, i)$, and j is a character of H. The double cosets decompose as

$$\begin{aligned} H\Delta(0,0)\Delta H &= H\Delta = H(0,0) + H(0,1)\,, \\ H\Delta(0,1)\Delta H &= H(0,0) + H(0,1)\,. \end{aligned}$$

Then α-induction gives:

$$\begin{aligned} &\alpha_{0,0} = [[0,0]]\,, \\ &\alpha^+_{0,1} = [[0,0]]\,, \quad \alpha^-_{0,1} = [[0,1]]\,, \\ &\alpha^+_{1,0} = [[1,0]]\,, \quad \alpha^-_{1,0} = [[0,0]]\,, \\ &\alpha^+_{1,1} = [[1,0]]\,, \quad \alpha^-_{1,1} = [[0,1]]\,. \end{aligned}$$

This gives the modular invariant $Q = xy^*$ and $Q^t = yx^*$.
(iv) *Example:* H_6 with nontrivial twist. This yields the permutation invariant $\mathcal{Z}_2 = \mathcal{Z}_3$.

4.2 $G = \mathbb{Z}_3$

Here there are 8 distinct modular invariants. All can be realised from subfactors or module categories. The following are the 6 subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_3$:

$$\begin{aligned} H_1 &= \{(0,0)\}\,, \\ H_2 &= \{(0,0),(1,2),(2,1)\}\,, \\ H_3 &= \{(0,0),(1,1),(2,2)\} = \Delta(\mathbb{Z}_3)\,, \\ H_4 &= \mathbb{Z}_3 \times \{0\}\,, \\ H_5 &= \{0\} \times \mathbb{Z}_3\,, \\ H_6 &= \mathbb{Z}_3 \times \mathbb{Z}_3.\,. \end{aligned}$$

In this case the cohomology $H^2(H,\mathbb{T})$ is nontrivial only for H_6 when it is $\mathbb{Z}_3$. Only H_1, H_2, H_3 and H_6 give rise to type I modular invariants or commutative Q-systems. As for $\mathbb{Z}_2$, the subgroups H_3 and H_6 give the modular invariants $\mathcal{Z}_1 = 1$ and $\mathcal{Z}_4 = xx^*$ respectively and H_1 gives $\mathcal{Z}_5 = yy^*$. The subgroups H_4 and H_5 give $\mathcal{Z}_7 = xy^*$ and $\mathcal{Z}_8 = yx^*$, whilst H_6 with its nontrivial twists from $H^2 = \mathbb{Z}_3$ give $\mathcal{Z}_2$ and $\mathcal{Z}_3$. The remaining subgroup H_2 yields the conjugation invariant $\mathcal{Z}_6$.

4.3 $G = S_3$

There are 48 distinct modular invariants but only 28 can be realised from subfactors from module categories. There are 22 distinct non-conjugate subgroups of $S_3 \times S_3$, which together with some non-trivial 2-cohomology is enough to produce all the 28 invariants. The following are the subgroups of $S_3 \times S_3$ which give rise to type I modular invariants or commutative Q-systems:

$$\begin{aligned} H_1 \;&= \{(1,1)\}\,, \\ H_4 \;&= \Delta(\mathbb{Z}_2)\,, \\ H_7 \;&= \Delta(\mathbb{Z}_3)\,, \\ H_8 \;&= \mathbb{Z}_2 \times \mathbb{Z}_2\,, \\ H_{11} &= \Delta(S_3)\,, \\ H_{14} &= \mathbb{Z}_3 \times \mathbb{Z}_3\,, \\ H_{19} &= K = \Delta(S_3)\cdot(1 \times \mathbb{Z}_3) \simeq (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2\,, \\ H_{20} &= S_3 \times S_3\,. \end{aligned}$$

The 2-cohomology $H^2(H,\mathbb{T})$ is non trivial only for $H_8, H_{14}, H_{19}, H_{20}$ in the above list as well as for $\mathbb{Z}_2 \times S_3$ and $S_3 \times \mathbb{Z}_2$.

The group S_3 is generated by σ and τ, with $\sigma^3 = \tau^2 = 1, \tau\sigma\tau = \sigma^2$. There are three Δ-Δ double cosets: $\Delta(1,1)\Delta, \Delta(\sigma,1)\Delta, \Delta(\tau,1)\Delta$, with stabilisers isomorphic to $S_3, \mathbb{Z}_3, \mathbb{Z}_2$, respectively. Denote the correspoding irreducible representations by $\{1,\varepsilon,\pi\}$, (where ε is the parity and π the two

dimensional representation), $\{1,\omega,\omega^2\}$,$\{1,\epsilon\}$, and we denote in this order the corresponding irreducible bundles as $\{0,1,2,3,4,5,6,7\}$ as usual as in [16, 25].
(i) *Example:* $H_{11} = \Delta(S_3)\,, H_{19} = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2\,, H_{20} = S_3 \times S_3\,.$
The subgroups H_{11}, H_{19}, H_{20} contain Δ and (in the case of the untwisted 2-cohomology) have corresponding modular invariants $1, Z_{(33)}, Z_{55}$ respectively:

$$\begin{aligned} Z_{(33)} &= |\chi_0+\chi_3|^2 + |\chi_1+\chi_3|^2 + |\chi_6|^2 + |\chi_7|^2\,, \\ Z_{55} &= |\chi_0+\chi_3+\chi_6|^2\,. \end{aligned}$$

Next consider the symmetric cases contained in Δ, namely H_1, H_4 and H_7 (where H^2 always vanishes).
(ii) *Example:* $H_1 = \{(1,1)\}\,.$
Take an irreducible bundle $[\Delta k\Delta,\psi]$ of $Bun^{\Delta-\Delta}(\Gamma)$, where ψ is an irreducible representation of the stabiliser Δ_k of the double coset $\Delta k\Delta$. Then, for the subgroup $H_1 = 0$, $\alpha^{\pm}$-induction is again $[\Delta(g,1)\Delta,\psi] \to \dim(\psi)[(\mathrm{g},1)]$ and $\dim(\psi)[(1,\mathrm{g})]$ respectively, so that the corresponding modular invariant is

$$\langle\alpha^+[\Delta(g,1)\Delta,\psi],\alpha^-[\Delta(h,1)\Delta,\chi]\rangle = \dim(\psi)\dim(\chi)[\delta_{\mathrm{g},1},\delta_{\mathrm{h},1}],$$

i.e. the mass matrix Z_{22}:

$$Z_{22} = |\chi_0+\chi_1+2\chi_2|^2$$

There are two further subgroups contained in Δ, namely $H_4 = \Delta(\mathbb{Z}_2)$ and $H_7 = \Delta(\mathbb{Z}_3)$.
(iii) *Example:* $H_4 = \Delta(\mathbb{Z}_2)\,.$
Consider first the case $H = H_4$, which has ten double cosets:

$$\begin{aligned} &H(1,1)H = H, H(1,\sigma)H, H(\sigma,1)H, H(\sigma,\sigma)H, H(\sigma,\sigma^2)H, \\ &H(1,\sigma\tau)H, H(\sigma\tau,1)H, H(\sigma\tau,\sigma)H, H(\sigma\tau,\sigma^2)H, H(1,\tau)H\,, \end{aligned}$$

with corresponding stabilisers $\mathbb{Z}_2$ for the first and last listed cosets and 1 for the remaining eight cosets, giving twelve irreducible bundles in $Bun^{H-H}(\Gamma)$. We decompose as H-H double cosets:

$$\begin{aligned} H\Delta(1,1)\Delta H &= H(1,1)H + H(\sigma,\sigma)H\,, \\ H\Delta(\sigma,1)\Delta H &= H(\sigma,1)H + H(\sigma,\sigma^2)H + H(1,\sigma)H\,, \\ H\Delta(\tau,1)\Delta H &= H(\tau,1)H + H(\sigma\tau,1)H + H(1,\sigma\tau)H \\ &+\;\; H(\sigma,\sigma\tau)H + H(\sigma\tau,\sigma^2)H\,. \end{aligned}$$

Then α-induction becomes:

$$\begin{aligned}
\alpha_{1,1} &= [(1,1),0]\,, \\
\alpha^{\pm}_{1,\epsilon} &= [(1,1),1]\,, \\
\alpha^{\pm}_{1,\pi} &= [(1,1),0] + [(1,1),1]\,, \\
\alpha^{\pm}_{(\sigma,1),1} &= [(\sigma,1),0]\,, \quad [(1,\sigma),0]\,, \\
\alpha^{\pm}_{(\sigma,1),\omega} &= [(\sigma,1),0]\,, \quad [(1,\sigma),0]\,, \\
\alpha^{\pm}_{(\sigma,1),\omega^2} &= [(\sigma,1),0]\,, \quad [(1,\sigma),0]\,, \\
\alpha^{\pm}_{(\tau,1),1} &= [(\tau,1),0] + [(\sigma\tau,1),0]\,, \quad [(1,\tau),0] + [(1,\sigma\tau),0]\,, \\
\alpha^{\pm}_{(\tau,1),\epsilon} &= [(\tau,1),1] + [(\sigma\tau,1),0]\,, \quad [(1,\tau),1] + [(1,\sigma\tau),0]\,.
\end{aligned}$$

Consequently, we have the irreducible objects $[\alpha_0^{\pm}]$, $[\alpha_1^{\pm}]$, $[\alpha_2^{\pm}] = [\alpha_0] \oplus [\alpha_1^{\pm}]$, $[\alpha_3^{\pm}] = [\alpha_4^{\pm}] = [\alpha_5^{\pm}]$, $[\alpha_6^{\pm}] = [\alpha_6^{\pm(1)}] \oplus [\alpha_6^{\pm(2)}]$ and $[\alpha_7^{\pm}] = [\alpha_6^{\pm(1)}] \oplus [\alpha_7^{\pm(2)}]$. Moreover, $[\alpha_1^{+}] = [\alpha_1^{-}]$ denoted henceforth by $[\alpha_1]$, $[\alpha_6^{+(2)}] = [\alpha_6^{-(2)}]$ denoted from now by $[\alpha_6^{(2)}]$ and similarly $[\alpha_7^{(2)}]$. Thus ${}_M\mathcal{X}_M^0 = \{\alpha_0, \alpha_1, \alpha_6^{(2)}, \alpha_7^{(2)}\}$. Also the $\mathcal{X}^{\pm}$-chiral systems are ${}_M\mathcal{X}_M^{\pm} = \{\alpha_0, \alpha_1, \alpha_6^{(2)}, \alpha_7^{(2)}, \alpha_3^{\pm}, \alpha_6^{\pm(1)}\}$. The corresponding modular invariant is $Z_{(22)}$:

$$Z_{(22)} = |\chi_0 + \chi_2|^2 + |\chi_1 + \chi_2|^2 + |\chi_6|^2 + |\chi_7|^2$$

(iv) *Example:* $H_7 = \Delta(\mathbb{Z}_3)$.
The next case $H = H_7$ has 8 double cosets:

$$\begin{aligned}
&H(1,1) = H, H(1,\sigma)H = H(1,\sigma), H(1,\sigma^2)H = H(1,\sigma^2), H(1,\tau)H = (\mathbb{Z}_3 \times \mathbb{Z}_3)(1,\tau), \\
&\qquad H(\tau,1)H = (\mathbb{Z}_3 \times \mathbb{Z}_3)(\tau,1), H(\tau,\tau)H = H(\tau,\tau), H(\sigma\tau,\tau)H, H(\sigma^2\tau,\tau)H\,,
\end{aligned}$$

with corresponding stabilisers $\mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3, 1, 1, \mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_3$ respectively giving twenty irreducible objects in $Bun^{H_7 - H_7}(\Gamma)$. We decompose as H-H double cosets:

$$\begin{aligned}
H\Delta(1,1)\Delta H &= H(1,1)H + H(\tau,\tau)H\,, \\
H\Delta(\sigma,1)\Delta H &= H(1,\sigma)H + H(\tau,\sigma\tau)H + H(\sigma\tau,\tau)H + H(1,\sigma^2)H\,, \\
H\Delta(\tau,1)\Delta H &= H(1,\tau)H + H(\tau,1)H\,.
\end{aligned}$$

Then α-induction becomes:

$$\begin{aligned}
\alpha_{1,1} &= [(1,1),0)]\,, \\
\alpha^{\pm}_{1,\epsilon} &= [(1,1),0)]\,, \\
\alpha^{\pm}_{1,\pi} &= [(1,1),\omega)] + [(1,1),\omega^2)]\,, \\
\alpha^{\pm}_{(\sigma,1),1} &= [(\sigma,1),1)] + [(1,\sigma),1)]\,, \\
\alpha^{\pm}_{(\sigma,1),\omega} &= [(\sigma,1),\omega)] + [(1,\sigma),\omega^2)]\,, \\
\alpha^{\pm}_{(\sigma,1),\omega^2} &= [(\sigma,1),\omega^2)] + [(1,\sigma),\omega)]\,, \\
\alpha^{\pm}_{(\tau,1),1} &= [(\tau,1),1)]\,, \quad [(1,\tau),1]\,, \\
\alpha^{\pm}_{(\tau,1),\epsilon} &= [(\tau,1),1)]\,, \quad [(1,\tau),1]\,.
\end{aligned}$$

Consequently, we have the irreducible objects: $[\alpha_0^\pm] = [\alpha_1^\pm]$, $[\alpha_2^\pm] = [\alpha_2^{\pm(1)}] \oplus [\alpha_2^{\pm(2)}]$, $[\alpha_3^\pm] = [\alpha_3^{\pm(1)}] \oplus [\alpha_3^{\pm(2)}]$, $[\alpha_4^\pm] = [\alpha_4^{\pm(1)}] \oplus [\alpha_4^{\pm(2)}]$, $[\alpha_5^\pm] = [\alpha_5^{\pm(1)}] \oplus [\alpha_5^{\pm(2)}]$, $[\alpha_6^\pm] = [\alpha_7^\pm]$, with $[\alpha_0^\pm], [\alpha_j^{\pm(i)}]$ and $[\alpha_6^\pm]$ irreducible sectors ($i = 1, 2; j = 2, 3, 4, 5$). The commutative neutral system as sectors is formed with nine automorphisms $[\alpha_0], [\alpha_j^{(i)}]$, with $i = 1, 2; j = 2, 3, 4, 5$ isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. Hence the system ${}_M\mathcal{X}_M^\pm = {}_M\mathcal{X}_M^0 \cup \{\alpha_6^\pm\}$, with the other fusion rules given by $[\alpha_j^{(i)}][\alpha_6^\pm] = [\alpha_6^\pm][\alpha_j^{(i)}] = [\alpha_6^\pm], i = 1, 2; j = 2, 3, 4, 5$. The other nine irreducible objects are from the decomposition:

$$\alpha_6^+ \alpha_6^- = \sum_{i,j} [(\tau, \sigma^i \tau), \omega^j] \,.$$

The corresponding modular invariant is:

$$Z_5 = |\chi_0 + \chi_1|^2 + 2|\chi_2|^2 + 2|\chi_3|^2 + 2|\chi_4|^2 + 2|\chi_5|^2 \,.$$

There are two further symmetric groups $H_8 = \mathbb{Z}_2 \times \mathbb{Z}_2$, and $H_{14} = \mathbb{Z}_3 \times \mathbb{Z}_3$. (vi) *Example:* $H_8 = \mathbb{Z}_2 \times \mathbb{Z}_2$. Here with $H = H_8$, there are four double cosets:

$$H(1,1)H = H, H(\sigma,1)H, H(1,\sigma)H, H(\sigma,\sigma)H$$

with stabilisers $\mathbb{Z}_2 \times \mathbb{Z}_2$, $1 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times 1$, 1 respectively so that there are nine irreducible objects in $Bun^{H-H}(\Gamma)$. We decompose as H-H double cosets:

$$\begin{aligned}
H\Delta(1,1)\Delta H &= H(1,1)H + H(\sigma,\sigma)H \,, \\
H\Delta(\sigma,1)\Delta H &= H(1,\sigma)H + H(\sigma,1) + H(\sigma,\sigma)H \,, \\
H\Delta(\tau,1)\Delta H &= H(1,1)H + H(1,\sigma)H + H(\sigma,1) + H(\sigma,\sigma)H \,.
\end{aligned}$$

Then α-induction becomes:

$$\begin{aligned}
\alpha_{1,1} &= [(1,1),0)] \,, \\
\alpha_{1,\epsilon}^\pm &= [(1,1),(1,0))] \,, \quad [(1,1),(0,1))] \,, \\
\alpha_{1,\pi}^\pm &= [(1,1),0)] + [(1,1),(1,0)] \,, \quad [(1,1),0)] + [(1,1),(0,1)] \,, \\
\alpha_{(\sigma,1),1}^\pm &= [(\sigma,1),(0,0)] \,, \quad [(1,\sigma),(0,0)] \,, \\
\alpha_{(\sigma,1),\omega}^\pm &= [(\sigma,1),(0,0)] \,, \quad [(1,\sigma),(0,0)] \,, \\
\alpha_{(\sigma,1),\omega^2}^\pm &= [(\sigma,1),(0,0)] \,, \quad [(1,\sigma),(0,0)] \,, \\
\alpha_{(\tau,1),1}^\pm &= [(1,1),0] + [(\sigma,1),0] \,, [(1,1),0] + [(1,\sigma),0] \,, \\
\alpha_{(\tau,1),\epsilon}^\pm &= [(1,1),(1,0)] + [(\sigma,1),(1,0)] \,, [(1,1),(0,1)] + [(1,\sigma),(0,0)] \,.
\end{aligned}$$

So computing we get that ${}_M\mathcal{X}_M^\pm = \{\alpha_0, \alpha_1^\pm, \alpha_3^\pm\}$ with $[\alpha_2^\pm] = [\alpha_0] \oplus [\alpha_1^\pm]$, $[\alpha_5^\pm] = [\alpha_4^\pm] = [\alpha_3^\pm]$, $[\alpha_6^\pm] = [\alpha_0] \oplus [\alpha_3^\pm], [\alpha_7^\pm] = [\alpha_1^\pm] \oplus [\alpha_3^\pm]$. The sectors of ${}_M\mathcal{X}_M^\pm$ are $\widehat{S_3}$. The corresponding modular invariant is

$$Z_{44} = |\chi_0 + \chi_2 + \chi_6|^2 \,.$$

(vi) *Example:* $H = H_{14} = \mathbb{Z}_3 \times \mathbb{Z}_3$.

Taking $H = H_{14}$, we have four double cosets:

$$H(1,1)H = H\,, H(\tau,1)H\,, H(1,\tau)H, H(\tau,\tau)H\,,$$

all with stabilisers H so that there are 36 irreducible objects in $Bun^{H-H}(\Gamma)$. We decompose as H-H double cosets:

$$\begin{aligned} H\Delta(1,1)\Delta H &= H(1,1)H + H(\tau,\tau)H\,,\\ H\Delta(\sigma,1)\Delta H &= H(1,1)H + H(\tau,\tau)H\,,\\ H\Delta(\tau,1)\Delta H &= H(1,\tau)H + H(\tau,1)H\,. \end{aligned}$$

Then α-induction becomes:

$$\begin{aligned}
\alpha_{1,1} &= [(1,1),1)]\,,\\
\alpha^{\pm}_{1,\epsilon} &= [(1,1),1]\,,\\
\alpha^{\pm}_{1,\pi} &= 2[(1,1),1]\,,\\
\alpha^{\pm}_{(\sigma,1),1} &= [(1,1),(\omega,1)] + [(1,1),(\omega^2,1)]\,, \quad [(1,1),(1,\omega)] + [(1,1),(1,\omega^2)]\,,\\
\alpha^{\pm}_{(\sigma,1),\omega} &= [(1,1),(\omega,1)] + [(1,1),(\omega^2,1)]\,, \quad [(1,1),(1,\omega)] + [(1,1),(1,\omega^2)]\,,\\
\alpha^{\pm}_{(\sigma,1),\omega^2} &= [(1,1),(\omega,1)] + [(1,1),(\omega^2,1)]\,, \quad [(1,1),(1,\omega)] + [(1,1),(1,\omega^2)]\,,\\
\alpha^{+}_{(\tau,1),1} &= [(\tau,1),(1,1)] + [(\tau,1),(\omega,1)] + [(\tau,1),(\omega^2,1)]\,,\\
\alpha^{-}_{(\tau,1),1} &= [(1,\tau),(1,1)] + [(1,\tau),(1,\omega)] + [(1,\tau),(1,\omega^2)]\,,\\
\alpha^{+}_{(\tau,1),\epsilon} &= [(\tau,1),(1,1)] + [(\tau,1),(\omega,1)] + [(\tau,1),(\omega^2,1)]\,,\\
\alpha^{-}_{(\tau,1),\epsilon} &= [(1,\tau),(1,1)] + [(1,\tau),(1,\omega)] + [(1,\tau),(1,\omega^2)]\,.
\end{aligned}$$

Hence we have the chiral system:

$${}_M\mathcal{X}^{+}_M = \{\alpha_0, \alpha_3^{+(1)}, \alpha_3^{+(2)}, \alpha_6^{+(1)}, \alpha_6^{+(2)}, \alpha_6^{+(3)}\}$$

with $[\alpha_0] = [\alpha_1^+]$, $[\alpha_2^+] = 2[\alpha_0]$, $[\alpha_3^+] = [\alpha_4^+] = [\alpha_5^+] = [\alpha_3^{+(1)}] \oplus [\alpha_3^{+(2)}]$, $[\alpha_7^+] = [\alpha_6^+] = [\alpha_6^{+(1)}] \oplus [\alpha_6^{+(2)}] \oplus [\alpha_6^{+(3)}]$, and similarly

$${}_M\mathcal{X}^{-}_M = \{\alpha_0, \alpha_3^{-(1)}, \alpha_3^{-(2)}, \alpha_6^{-(1)}, \alpha_6^{-(2)}, \alpha_6^{-(3)}\}\,.$$

We can conclude that ${}_M\mathcal{X}^{\pm}_M$ is as sectors S_3. The corresponding modular invariant is Z_{33}:

$$Z_{33} = |\chi_0 + \chi_1 + 2\chi_3|^2\,.$$

Acknowledgement. This work was supported by the EU Network QSNG in Quantum Spaces - Noncommutative Geometry and the EPSRC Network ABC-KLM in Algebraic Geometry, Boundary Conformal Field Theory and Noncommutative Geometry. I thank Daniel Hoyt for assistance in some of the computations and discusssions at an early stage of this work, Ryszard Nest for discussions throughout and to Paulo Pinto for a careful reading of the manuscript. I am grateful to ANU Canberra, Copenhagen University, DIAS Dublin, IST Lisbon, Oxford University, Tor Vergata Rome, the University of Oslo and particularly Alan Carey, Ryszard, Nest, Denjoe O'Connor, Paulo Pinto, Keith Hannabuss, Roberto Longo, John Roberts, Ola Bratteli, Erling Stormer for their warm hospitality during visits when much of this work was done.

References

1. Altschuler, A., Coste, A.: Quasi-quantum groups, knots, three manifolds and topological field theory. Commun. Math. Phys., **150**, 83–107 (1992).
2. Altschuler, A., Coste, A., Maillard, J-M.: Representation theory of twisted group double. Ann. Fond. Louis de Broglie. **29**, 681–694 (2004).
3. Atiyah, M.F.: K-theory past and present. Sitzungsberichte der Berliner Mathematischen Gesellschaft, 411–417, Berliner Math. Gesellschaft, Berlin, 2001. math.KT/0012213
4. Atiyah, M.F., Segal, G.B.: Twisted K-theory. math.KT/0407054
5. Bakalov. B., Kirillov, A.: Lectures on tensor cateogries and modular functors. University Lecture Series, vol 21, American Mathematical Society, Providence, R.I. 2001
6. Baumgärtel, H.: *Operatoralgebraic methods in quantum field theory.* Berlin: Akademie Verlag, 1995
7. Böckenhauer, J., Evans, D.E.: Modular invariants, graphs and α-induction for nets of subfactors. I. Commun. Math. Phys., **197**, 361–386 (1998); II Commun. Math. Phys., **200**, 57–103 (1999); III Commun. Math. Phys. **205**, 183–228 (1999)
8. Böckenhauer, J., Evans, D.E.: Modular invariants from subfactors: Type I coupling matrices and intermediate subfactors. Commun. Math. Phys., **213**, 267–289 (2000)
9. Böckenhauer, J., Evans, D.E.: *Modular invariants from subfactors*, in *Quantum symmetries in theoretical physics and mathematics* (Bariloche, 2000), pp 95–131, Contemp. Math., 294, Amer. Math. Soc., Providence, RI, 2002
10. Böckenhauer, J., Evans, D.E.: Modular invariants and subfactors. Mathematical physics in mathematics and physics (Siena, 2000), 11–37, Fields Inst. Commun., 30, Amer. Math. Soc., 2001.
11. Böckenhauer, J., Evans, D.E., Kawahigashi, Y.: On α-induction, chiral generators and modular invariants for subfactors. Commun. Math. Phys., **208**, 429–487 (1999)
12. Böckenhauer, J., Evans, D.E., Kawahigashi, Y.: Chiral structure of modular invariants for subfactors. Commun. Math. Phys., **210**, 733-784 (2000)
13. Böckenhauer, J., Evans, D.E., Kawahigashi, Y.: Longo-Rehren subfactors arising from α-induction. Publ. RIMS, Kyoto Univ., **37**, 1–35 (2001)
14. Bouwknegt, P., Carey, A.L., Mathai, V., Murray, M.M., Stevenson, D.: Twisted K-theory and K-theory of bundle gerbes. hep-th/0106194
15. Cardy, J.: Fusion rules and the Verlinde formula. Nucl. Phys. **B**, **324**, 581–596 (1989)
16. Coste, A., Gannon, T., Philippe, R.: Finite group modular data. Nucl. Phys. **B**, **581**, 679–717 (2000)
17. Dijkgraaf, R.; Cumrun, V.; Verlinde, E.; Verlinde, H.: The operator algebra of orbifold models. Commun. Math. Phys., **123**, 485-526 (1989)
18. Dijkgraaf, R., Pasquier, V., Roche, P.: Quasi-quantum groups related to orbifolds models. Modern quantum field theory (Bombay, 1990), 375–383, World Sci. Publishing, River Edge, NJ, 1991.
19. Dixmier, J., Douady, A.: Champs continus d'espaces hilbertiens et de C*-algebras. Bull. Soc. Math. France, **91**, 227-284 (1963)
20. Evans, D.E.: Fusion rules of modular invariants. Rev. Math. Phys., **14**, 709-732 (2002)

21. Evans, D.E.: Critical phenomena, modular invariants and operator algebras. Operator algebras and mathematical physics (Constanţa 2001, J. Cuntz, G.A. Elliott, S. Stratila et al. eds.), The Theta Foundation, Bucharest, 2003
22. Evans, D.E.: Modular invariant partition functions in statistical mechanics, conformal field theory and their realisation by subfactors. Proceedings Congress of International Association of Mathematical Physics (Lisbon 2003, J-C Zambrini, eds.), pp. 464–475, World Scientific Press. Singapore, 2005.
23. Evans, D. E., Kawahigashi, Y.: Quantum symmetries on operator algebras. Oxford University Press (1998)
24. Evans, D.E., Pinto, P.R.: Modular invariants and their fusion rules in *Advances in Quantum Dynamics* (Mount Holyoke, 2002), pp. 119–130, Contemp. Math. **335**, Amer. Math. Soc., Providence, RI, 2003
25. Evans, D.E., Pinto, P.R.: Subfactor realisation of modular invariants. Commun. Math. Phys., **237**, 309–363 (2003)
26. Evans, D.E., Pinto, P.R.: Modular invariants and the double of the Haagerup subfactor. In *Advances in Operator Algebras and Mathematical Physics* (Sinaia 2003, F.-P. Boca, O. Bratteli, R. Longo, H. Siedentop eds.), pp. 67–88, The Theta Foundation, Bucharest, 2006.
27. Evans, D.E., Pinto, P.R.: Subfactor realisation of modular invariants, II.
28. Freed, D.S.: Higher algebraic structures and quantisation. Commun. Math. Phys., **159**, 343–398 (1994).
29. Freed, D.S.: The Verlinde algebra is twisted equivariant K-theory. Turkish J. Math., **25**, 159–167 (2001).
30. Freed, D.S.: Twisted K-theory and loop groups. Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), 419–430, Higher Ed. Press, Beijing, 2002.
31. Freed, D.S.: K-theory in quantum field theory. Current developments in mathematics, 2001, 41–87, Int. Press, Somerville, MA, 2002.
32. Freed, D.S., Hopkins, M.J., Teleman, C.: Twisted equivariant K-theory with complex coefficients. math-ph/020631
33. Freed, D.S., Quinn, F.: Chern Simmons theory with finite gauge group. Commun. Math. Phys., **156**, 435–472 (1993)
34. di Francesco, P., Zuber, J.B.: $SU(N)$ lattice integrable models associated with graphs. Nucl. Phys. **B**, **338**, 602–646 (1990)
35. Fröhlich, J., Gabbiani, F.: Operator algebras and conformal field theory. Commun. Math. Phys., **155**, 569–640 (1993)
36. Gannon, T.: Modular data: the algebraic combinatorics of conformal field theory. hep-th/0103044
37. Izumi, M.: The structure of sectors associated with Longo-Rehren inclusions, I. General theory. Commun. Math. Phys., **213**, 127–179 (2000)
38. Izumi, M.: The structure of sectors associated with Longo-Rehren inclusions, II. Examples. Rev. Math. Phys., **13**, 603–674 (2001)
39. Izumi, M., Kosaki, H.: On the subfactor analogue of the second cohomology. Preprint, Kyoto-Math 2002-01
40. Jones, V. F. R.: An invariant for group actions. Algèbres d'opèrateurs (Sèm., Les Plans-sur-Bex, 1978), pp. 237–253, Lecture Notes in Math., 725, Springer, Berlin, 1979.
41. Jones, V.F.R.: Index for subfactors, Invent. Math., **72**, 1–25 (1983).

42. Katayama, Y., Takesaki, M.: Outer actions of a countable discrete amenable group on an AFD factor. Advances in quantum dynamics (South Hadley, MA, 2002), 163–171, Contemp. Math., 335, Amer. Math. Soc., Providence, RI, 2003.
43. Kawahigashi, Y., Sato, N., Wakui, M. (2+1)-dimensional topological quantum field theory from subfactors and Dehn surgery formula for 3-manifold invariants. Preprint math.OA/0208238
44. Kirillov, A., Jr.; Ostrik, V.: On a q-analogue of the McKay correspondence and the ADE classification of $\hat{\mathfrak{sl}}_2$ conformal field theories. Adv. Math., **171**, 183-227 (2002)
45. Kosaki, H., Yamagami, S.: Irreducible bimodules associated with crossed product algebras. Internat. J. Math., **3**, 661–676 (1992)
46. Kosaki, H., Munemasa, A., Yamagami, S.: On fusion algebras associated to finite group actions. Pacific J. Math., **177**, 269–290 (1997)
47. Longo, R.: Index of subfactors and statistics of quantum fields, II. Commun. Math. Phys., **130**, 285–309 (1990)
48. Longo, R., Rehren, K.-H.: Nets of subfactors. Rev. Math. Phys., **7**, 567–597 (1995)
49. Mason, G.: The quantum double of a finite group and its role in conformal field theory. Groups '93 Galway/St. Andrews, Vol. 2, 405–417, London Math. Soc. Lecture Note Ser., **212**, Cambridge Univ. Press, Cambridge, 1995
50. Ocneanu, A.: Paths on Coxeter diagrams: From Platonic solids and singularities to minimal models and subfactors, in *Lectures on operator theory,* Rajarama Bhat, B.V. et al. (eds.) (Notes recorded by S. Goto), The Fields Institute Monographs, Providence, Rhode Island: AMS publications 2000, pp. 243–323
51. Ocneanu, A.: The classification of subgroups of quantum $SU(N)$, in *Quantum symmetries in theoretical physics and mathematics (Bariloche, 2000)*, Contemp. Math., 294, Amer. Math. Soc., Providence, RI, 2002, pp 133–159
52. Ostrik, V.: Module categories, weak Hopf algebras and modular invariants. Transform. Groups, **8**, 177– 206 (2003) or math.QA/0111139
53. Ostrik, V.: Module categories for quantum doubles of finite groups. Inter. Math. Research. Notices, **27**, 1507–1520 (2003), math.QA/0202130
54. Rieffel, M.: Morita equivalence for operator algebras. Operator algebras and applications, Part I (Kingston, Ont., 1980), pp. 285–298, Proc. Sympos. Pure Math., 38, Amer. Math. Soc., Providence, R.I., 1982.
55. Sato, N., Wakui, M.: Computations of Turaev-Viro-Ocneanu invariants of 3-manifolds from subfactors. J. Knot Theory Ramifications, **12**, 543–574 (2003)
56. Sutherland, C.: Cohomology and extensions of von Neumann algebras. I, II. Publ. RIMS, Kyoto Univ., **16**, 105–133, 135–174 (1980)
57. Turaev, V.G. Quantum invariants of knots and 3-manifolds. in *de Gruyter Studies in Mathematics*, vol 18. Walter de Gruyter, Berlin, 1994
58. Verlinde, E.: Fusion rules and modular transformations in 2D conformal field theory. Nucl. Phys. **B**, **300**, 360-376 (1988)
59. Wassermann, A.: Operator algebras and conformal field theory III: fusion of positive representations of LSU(n) using bounded operators. Invent. Math., **33**, 467–538 (1998)
60. Wassermann, A.: Automorphic represenations of real and p-adic Lie groups. Noncommutative Geometry, Oberwolfach Report No. 45/2004.
61. Xu, F.: New braided endomorphisms from conformal inclusions. Commun. Math. Phys., **192**, 349–403 (1998)

The Orbit Structure of Cantor Minimal $\mathbb{Z}^2$-Systems

Thierry Giordano[1] *, Ian F. Putnam[2] †, and Christian F. Skau[3] ‡

[1] Department of Mathematics and Statistics, University of Ottawa, 585 King Edward Ave., Ottawa, Ontario, Canada K1N 6N5, giordano@uottawa.ca.
[2] Department of Mathematics and Statistics, University of Victoria, Victoria, B.C., Canada V8W 3P4, putnam@math.uvic.ca.
[3] Norwegian University of Science and Technology (NTNU), N-7491 Trondheim, Norway, csk@math.ntnu.no.

1 Introduction

In 1959, H. Dye ([D1]) introduced the notion of orbit equivalence and proved that any two ergodic finite measure preserving transformations on a Lebesgue space are orbit equivalent. In [D2], he had also conjectured that an arbitrary ergodic action of a discrete amenable group is orbit equivalent to a $\mathbb{Z}$-action. This conjecture was proved by Ornstein and Weiss in [OW]. The most general case was proved by Connes, Feldman and Weiss ([CFW]) by establishing that an amenable non-singular countable equivalence relation $\mathcal{R}$ can be generated by a single transformation, or equivalently, is hyperfinite, i.e., $\mathcal{R}$ is up to a null set, a countable increasing union of finite equivalence relations.

For the Borel case, Weiss ([W]) proved that actions of $\mathbb{Z}^n$ are (orbit equivalent to) hyperfinite Borel equivalence relations, whose classification was obtained by Dougherty, Jackson and Kechris ([DJK]). It is not yet known if an arbitrary Borel action of a discrete amenable group is orbit equivalent to a $\mathbb{Z}$-action.

Our main interest in this report is the case of a free minimal continuous action φ of $\mathbb{Z}^2$ on a Cantor set (i.e., a compact totally disconnected metric space with no isolated points). However, let us begin with a more general group action and consider a free action φ of a countable discrete group on a compact metric space X (i.e., for every $g \in G$, $\varphi(g) \in \mathrm{Homeo}(X)$, and $\varphi(g)x = x$ for some $x \in X$ if and only if $g = id$). Recall that the action φ is *minimal* if the φ-orbit of every point of X is dense in X.

*Research supported in part by operating grants from NSERC (Canada).
†Research supported in part by operating grants from NSERC (Canada).
‡Research supported in part by the Norwegian Research Council for Science and Humanities.

Given two free group actions (X, G, φ) and (Y, H, ψ), an *isomorphism* between them is a homeomorphism $h : X \to Y$ and a group isomorphism $\alpha : G \to H$ such that, for all $g \in G$, we have

$$h \circ \varphi(g) = \psi(\alpha(g)) \circ h \ .$$

Recall from [GPS2] that, given two free group actions (X, G, φ) and (Y, H, ψ), an *orbit equivalence* between them is a homeomorphism $h : X \to Y$ such that, for every $x \in X$, we have

$$h(\mathcal{O}_\varphi(x)) = \mathcal{O}_\psi(h(x)) \,,$$

where $\mathcal{O}_\varphi(x)$ denotes the orbit of the point $x \in X$ under the action of φ. It is clear from the definitions that every isomorphism is also an orbit equivalence.

For connected spaces, using a result of Sierpinski (see [K], Thm 6, Ch V, 47, III), any orbit equivalence is also an isomorphism. Therefore, we will consider only spaces which are totally disconnected.

2 Étale Equivalence Relations

Let X be a compact metric space and G be a countable group with the discrete topology. If φ is a free continuous action of G on X, let $\mathcal{R}_\varphi$ denote the equivalence relation given by

$$\mathcal{R}_\varphi = \{(x, \varphi(g)x) \,;\ x \in X,\ g \in G\} \ .$$

With the product topology, $X \times G$ is a σ-compact, locally compact space; then using the bijection from $X \times G$ to $\mathcal{R}_\varphi$ given by $(x, g) \mapsto (x, \varphi(g)x)$, the equivalence relation $\mathcal{R}_\varphi$ becomes a topological groupoid. If r and s (for range and source) denote the two canonical projections from $\mathcal{R}_\varphi$ to X:

$$s(x, \varphi(g)x) = \varphi(g)x \quad \text{and} \quad r(x, \varphi(g)x) = x \,,$$

then r and s are local homeomorphisms. Moreover as G is countable, each $\mathcal{R}_\varphi$ is a countable equivalence relation, i.e. each equivalence class $[x]_{\mathcal{R}_\varphi} = \{y \in X \mid (x, y) \in \mathcal{R}_\varphi\}$ is countable for each $x \in X$. Then $\mathcal{R}_\varphi$ is the motivating example of an étale equivalence relation, whose precise definition is as follows:

Definition 1. *The locally compact groupoid $(\mathcal{R}, \mathcal{T})$, where $\mathcal{R}$ is a countable equivalence relation on a compact metric space X, is étale if the maps $r, s : \mathcal{R} \to X$ are local homeomorphisms, i.e. for every $(x, y) \in \mathcal{R}$ there exists an open neighborhood $U \in \mathcal{T}$ of (x, y) so that $r(U)$ and $s(U)$ are open in X and $r : U \to r(U)$ and $s : U \to s(U)$ are homeomorphisms. If X is zero-dimensional, we may clearly choose U to be a clopen set.*

We will call $(\mathcal{R}, \mathcal{T})$ an étale equivalence relation on X.

Remark 2.
a) This definition is equivalent to the various definitions of an étale (or r-discrete) locally compact groupoid (applied in our setting) that can be found in the literature (see for example [Pa], [R]).
b) If $\mathcal{R}$ is an étale equivalence relation, then its equivalence classes are countable. By definition, $\mathcal{R}$ can be written as a union of graphs of local homeomorphisms of the form $s \circ r^{-1}$.
c) The topology $\mathcal{T}$ on $\mathcal{R}$ is rarely the relative topology from $\mathcal{R} \subset X \times X$. Indeed if $\mathcal{R}$ is étale and has an infinite equivalence class, then $\mathcal{T}$ is not the relative topology of $X \times X$.
d) A countable equivalence relation $\mathcal{R}$ on X may be given distinct non-isomorphic topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ so that $(\mathcal{R}, \mathcal{T}_1)$ and $(\mathcal{R}, \mathcal{T}_2)$ are étale equivalence relations. This contrasts with the situation in the countable (standard) Borel equivalence relation setting, where the Borel structure is uniquely determined by $\mathcal{R} \subset X \times X$.

Generalizing the statement and the proof of Theorem 1 of [FM], we have:

Proposition 3. *Let $(\mathcal{R}, \mathcal{T})$ be an étale equivalence relation on the zero-dimensional space X. There exists a countable group G of homeomorphisms of X so that $\mathcal{R} = \mathcal{R}_G$, where $\mathcal{R}_G = \{(x, gx)\,;\ x \in X,\ g \in G\}$.*

Remark 4. In [HM], Hjorth and Molberg have recently shown that the group G in Proposition 3 cannot always be chosen acting freely.

There are two natural notions of equivalence between étale equivalence relations:

Definition 5. *(Isomorphism and orbit equivalence) Let $(\mathcal{R}_1, \mathcal{T}_1)$ and $(\mathcal{R}_2, \mathcal{T}_2)$ be two étale equivalence relations on X_1 and X_2 respectively.*
1. $(\mathcal{R}_1, \mathcal{T}_1)$ and $(\mathcal{R}_2, \mathcal{T}_2)$ are orbit equivalent if there exists a homeomorphism $F : X_1 \to X_2$ so that

$$(x, y) \in \mathcal{R}_1 \iff (F(x), F(y)) \in \mathcal{R}_2\,.$$

We call such a map F an orbit map.
2. $(\mathcal{R}_1, \mathcal{T}_1)$ and $(\mathcal{R}_2, \mathcal{T}_2)$ are isomorphic if there is an orbit map $F : X_1 \to X_2$ so that $F \times F : (\mathcal{R}_1, \mathcal{T}_1) \to (\mathcal{R}_2, \mathcal{T}_2)$ is a homeomorphism.

Observe that $(\mathcal{R}_1, \mathcal{T}_1)$ is orbit equivalent to $(\mathcal{R}_2, \mathcal{T}_2)$, via the orbit map F if and only if $F([x]_{\mathcal{R}_1}) = [F(x)]_{\mathcal{R}_2}$ for each $x \in X_1$. So F maps equivalence classes into equivalence classes.

There is a notion, introduced by J. Renault ([R1]), of an invariant probability measure for an étale equivalence relation $\mathcal{R} \subset X \times X$. A measure μ on X is $\mathcal{R}$-*invariant* if $\mu(r(U)) = \mu(s(U))$, for every open set $U \subset \mathcal{R}$ such that $r : U \to r(U)$ and $s : U \to s(U)$ are homeomorphisms. We will denote by $M(X, \mathcal{R})$ the compact convex cone of $\mathcal{R}$-invariant probability measures on

X. If $F : X_1 \to X_2$ is an orbit equivalence between two étale equivalence relations, then F induces a bijection between the two sets of invariant probability measures $M(X_1, \mathcal{R}_1)$ and $M(X_2, \mathcal{R}_2)$.

3 Invariants for Cantor Étale Equivalence Relation

To an étale equivalence relation $\mathcal{R}$ on the Cantor set X we associate two ordered groups which are invariants of isomorphism and orbit equivalence of $\mathcal{R}$.

By an ordered group, we mean a countable abelian group G with a subset G^+, called the positive cone, such that

$$(i)\ G^+ + G^+ \subset G^+ \ , \quad (ii)\ G^+ - G^+ = G \ , \quad (iii)\ G^+ \cap (-G^+) = \{0\} \ .$$

By an order unit for (G, G^+), we mean an element $u \in G^+$ such that for every $a \in G^+$, $nu - a \in G^+$, for some $n \geq 1$.

Let $C(X, \mathbb{Z})$ be the abelian group of continuous functions with values in $\mathbb{Z}$. We denote by $B(X, \mathcal{R})$ the (coboundary) subgroup of $C(X, \mathbb{Z})$ generated by the functions $\chi_{r(U)} - \chi_{s(U)}$, where U is a clopen subset of $\mathcal{R}$ on which r and s are local homeomorphisms.

We define $B_m(X, \mathcal{R})$ to be the subgroup of $C(X, \mathbb{Z})$ of all functions f such that $\int_X f \, d\mu = 0$, for all $\mu \in M(X, \mathcal{R})$. Note that if $M(X, \mathcal{R}) = \emptyset$, then $B_m(X, \mathcal{R}) = C(X, \mathbb{Z})$.

Definition 6. *Let $\mathcal{R}$ be an étale equivalence relation on the Cantor set X. We denote by*
i) $D(X, \mathcal{R}) = C(X, \mathbb{Z})/B(X, \mathcal{R})$ the ordered group whose positive cone and order unit u are

$$D(X, \mathcal{R})^+ = \{[f] \ ; \ f \in C(X, \mathbb{Z}) \, , \ f \geq 0 \} \quad \textit{and} \quad u = [1] \ .$$

ii) $D_m(X, \mathcal{R}) = C(X, \mathbb{Z})/B_m(X, \mathcal{R})$ the ordered group whose positive cone and order unit u are

$$D_m(X, \mathcal{R})^+ = \{[f] \ ; \ f \in C(X, \mathbb{Z}) \, , \ f \geq 0 \} \quad \textit{and} \quad u = [1] \ .$$

Remark 7.
i) $B(X, \mathcal{R})$ is a subset of $B_m(X, \mathcal{R})$ and $D_m(X, \mathcal{R})$ is a quotient of $D(X, \mathcal{R})$.
ii) If $\mathcal{R}_\varphi$ denotes the étale equivalence relation induced by a minimal homeomorphism φ of the Cantor set X, we have:

a) By [P], Thm 4.1 and [HPS], Cor. 6.3 , the triple $(D(X, \mathcal{R}_\varphi), D(X, \mathcal{R}_\varphi)^+, [1])$ is a simple, acyclic dimension group with (canonical) order unit. Moreover any simple, acyclic dimension group (G, G^+, u) where u is a distinguished order unit, can be realized as $(D(X, \mathcal{R}_\varphi), D(X, \mathcal{R}_\varphi)^+, [1])$ for a Cantor minimal system (X, φ).

b) $B_m(X, \mathcal{R}_\varphi)/B(X, \mathcal{R}_\varphi)$ is equal to the infinitesimal subgroup $\mathrm{Inf}(D(X, \mathcal{R}_\varphi))$ of $D(X, \mathcal{R}_\varphi)$ and $D(X, \mathcal{R}_\varphi)/\mathrm{Inf}(D(X, \mathcal{R}_\varphi))$ is naturally isomorphic to $D_m(X, \mathcal{R}_\varphi)$.

It is then easy to check that:

Proposition 8. *If $F : X_1 \to X_2$ is an orbit map between two étale equivalence relations $(X_1, \mathcal{R}_1, \mathcal{T}_1)$ and $(X_2, \mathcal{R}_2, \mathcal{T}_2)$, then it induces an order isomorphism preserving the order units from $D_m(X_1, \mathcal{R}_1)$ to $D_m(X_2, \mathcal{R}_2)$.*

Moreover if F implements an isomorphism between $(X_1, \mathcal{R}_1, \mathcal{T}_1)$ and $(X_2, \mathcal{R}_2, \mathcal{T}_2)$, then it induces an order isomorphism preserving the order units from $D(X_1, \mathcal{R}_1)$ to $D(X_2, \mathcal{R}_2)$.

4 AF-Equivalence Relations

The AF equivalence relations ([R1], [GPS2]) form one of the most important classes of étale equivalence relations. The terminology AF comes from C*-algebra theory and means approximately finite.

Definition 9. *An étale equivalence relation $\mathcal{R}$ on X is an AF-relation if X is a totally disconnected compact metrizable space and if there are*

$$\mathcal{R}_1 \subset \mathcal{R}_2 \subset \cdots$$

such that $\cup_n \mathcal{R}_n = \mathcal{R}$ and $\mathcal{R}_n \subset \mathcal{R}$ is a compact open subequivalence relation, for each $n \geq 1$.

Before giving examples of AF-equivalence relations, let us note that:

Proposition 10. *([GPS2], Thm 3.8). Let φ be a free action of a countable group G on a totally disconnected compact metric space X. The relation $\mathcal{R}_\varphi$ is an AF-equivalence relation if and only if the group G is locally finite.*

Let us describe the fundamental example of an AF-equivalence relation. We begin with a Bratteli diagram (see [HPS], [Ef]). It is a locally finite, infinite directed graph which consists of a vertex set V and an edge set E written as a countable disjoint union of non-empty finite sets:

$$V = V_0 \cup V_1 \cup V_2 \cup \cdots \quad \text{and} \quad E = E_1 \cup E_2 \cup E_3 \cup \cdots$$

Each edge e in E_n has a source $i(e)$ in V_{n-1}, and a range $f(e)$ in V_n. For simplicity we assume that V_0 consists of a single vertex and for every other vertex v, $f^{-1}\{v\}$ and $i^{-1}\{v\}$ are non-empty.

The space $X = X_{(V,E)} = \{e = (e_1, e_2, \cdots) \ ; \ e_n \in E_n \, , i(e_{n+1}) = f(e_n) \text{ for } n \geq 1\}$ is the space of infinite paths in the diagram. It is given the relative topology of the product space $\prod_n E_n$ and is therefore compact metrizable and zero dimensional. For each $N \geq 0$, let

$$\mathcal{R}_N = \{(e,f) \in X \times X \; ; \; e_n = f_n \text{ for all } n > N\}\,.$$

With the relative topology of the product $X \times X$, then $\mathcal{R}_N$ is a compact étale equivalence relation (hence each equivalence class is finite), and $\mathcal{R}_N$ is an open subset of $\mathcal{R}_{N+1}$, for all $N \geq 1$.

Let $\mathcal{R} = \bigcup_{N=0}^{\infty} \mathcal{R}_N$, and give $\mathcal{R}$ the inductive limit topology. This means that a sequence $\{(x_n, y_n)\}$ in $\mathcal{R}$ converges to (x,y) in $\mathcal{R}$ if and only if $\{x_n\}$ converges to x, $\{y_n\}$ converges to y (in X) and, for some N, (x_n, y_n) is in R_N for all but finitely many n. Then $\mathcal{R}$ is easily seen to be an étale equivalence relation which will be denoted by $AF(V,E)$.

We observe that if (V', E') is a *telescope* of (V,E), i.e. (V',E') is obtained from (V,E) by telescoping (V,E) to certain levels $0 < n_1 < n_2 < n_3 < \cdots$, then $AF\,(V,E)$ is isomorphic to $AF\,(V',E')$. In fact, there is a natural homeomorphism $\alpha : X_{(V,E)} \to X_{(V',E')}$, and α clearly implements the isomorphism, according to the description we have given of convergence in $AF\,(V,E)$, respectively $AF\,(V',E')$.

The Bratteli diagram (V,E) is *simple* if for each n there is an $m > n$ so that by telescoping the diagram between levels n and m, every vertex v in V_n is connected to every vertex w in V_m. It is a simple observation that (V,E) is simple if and only if every $AF\,(V,E)$-equivalence class is dense in $X_{(V,E)}$.

The above example is in fact the general case. Indeed, we have:

Theorem 11. *Let $\mathcal{R}$ be an AF-relation on a totally disconnected compact metrizable space X. Then there exists a Bratteli diagram (V,E) such that $\mathcal{R}$ is isomorphic to the AF-equivalence relation $AF(V,E)$ associated to (V,E).*

Furthermore, (V,E) is simple if and only if $\mathcal{R}$ is minimal (i.e. every $\mathcal{R}$-equivalence class is dense).

5 The Classification of AF-Equivalence Relations

For AF-equivalence relations the invariants introduced in Definition 3.1 have not only a well-known structure, but they also form complete sets of invariants of AF-equivalence relations up to isomorphism and in the minimal case up to orbit equivalence. Indeed we have:

Theorem 12. *(see [HPS]) For a Bratteli diagram (V,E) and the associated AF-equivalence relation $AF(V,E)$, the group $D(X_{(V,E)}, AF(V,E))$ is the dimension group of the Bratteli diagram (V,E). It is simple if and only if $AF(V,E)$ is minimal.*

Approximately finite dimensional C*-algebras were classified in 1976 by G. Elliott. Building on this result, Krieger proved in [Kr] the following:

Theorem 13. *For AF-equivalence relations* $(X, \mathcal{R})$*, the triple formed by the ordered group* $(D(X, \mathcal{R}), D(X, \mathcal{R})^+)$ *and the order unit* [1] *is a complete invariant for isomorphism.*

For minimal AF-relations, we then can get:

Theorem 14. *For AF-equivalence relations* $(X, \mathcal{R})$*, the triple formed by the dimension group* $(D_m(X, \mathcal{R}), D_m(X, \mathcal{R})^+)$ *and the order unit* [1] *is a complete invariant for orbit equivalence.*

Even if Theorem 14 appears in [GPS1] as a corollary of the classification up to orbit equivalence of minimal $\mathbb{Z}$-actions, a direct proof can be given and it is in fact more logical to do so. Note that the range of the invariant for orbit equivalence of AF-equivalence relations is the class of all simple, acyclic dimension groups whose the infinitesimal subgroup is trivial.

6 The Strategy for Orbit Equivalence Results

Let G be $\mathbb{Z}$ or $\mathbb{Z}^2$ and φ be a minimal, free G-action on the Cantor set X. As the classification of AF-equivalence relations up to orbit equivalence is known, it is sufficient to show that such an action is *affable* (i.e., orbit equivalent to an AF-equivalence relation). This will be achieved with the following two steps:

(1) If φ and G are as above, construct a minimal AF-subequivalence relation $\mathcal{R}$ of $\mathcal{R}_\varphi$, two closed "small" subsets Y_0 and Y_1 of X, and a homeomorphism $\alpha : Y_0 \to Y_1$ such that the equivalence relation $\mathcal{R} \vee$ Graph (α), generated by $\mathcal{R}$ and the graph of α is equal to $\mathcal{R}_\varphi$.

(2) Prove then that $\mathcal{R} \vee$ Graph (α) is orbit equivalent to $\mathcal{R}$. The second step means that a minimal AF-relation $\mathcal{R}$ can be enlarged "slightly" and stays AF, more precisely still be orbit equivalent to $\mathcal{R}$. We will present the precise statement in the next section.

The first step depends on the group G. We have a complete answer for $G = \mathbb{Z}$ and up to now only a partial one for $\mathbb{Z}^2$.

Remark 15. For $G = \mathbb{Z}$, this strategy was used by Dye in the measurable case using repetitively the Rohlin lemma to get the first step, with the small subsets Y_0 and Y_1 having measure zero. This can be extended to include amenable groups. The second step is then not necessary.

For Borel actions of $\mathbb{Z}^n$, Weiss ([W]) used also the same strategy. Contrary to the (finite invariant measure) measurable case, AF-relations are not unique. They were classified by Dougherty, Jackson and Kechris ([DJK]) and their complete invariant up to orbit equivalence is the cardinality of the set of their finite invariant ergodic measures.

7 The Absorption Theorem

The second step of our strategy for orbit equivalence results will be accomplished with Theorem 16. This result states precise sufficient conditions under which a minimal AF-relation can be enlarged and stay orbit equivalent to itself. For an étale equivalence relation $\mathcal{R}$ on X, let us recall some terminology: if Y is a closed subset of X, we say that:

(1) Y is $\mathcal{R}$-étale if $\mathcal{R}|_Y(= \mathcal{R} \cap (Y \times Y))$, with the relative topology, is an étale equivalence relation on Y.

(2) Y is a thin subset if $\mu(Y) = 0$ for every finite $\mathcal{R}$-invariant measure μ.

Theorem 16. *([GPS2], Thm 4.18). Let $\mathcal{R}$ be a minimal AF-equivalence relation on the Cantor set X and let Y_0, Y_1 be two closed $\mathcal{R}$-étale and thin subsets of X. Suppose $\mathcal{R} \cap (Y_0 \times Y_1) = \emptyset$ (and so in particular $Y_0 \times Y_1 = \emptyset$), and let $\alpha : Y_0 \to Y_1$ be a homeomorphism such that $\alpha \times \alpha : \mathcal{R}|_{Y_0} \to \mathcal{R}|_{Y_1}$ is an isomorphism.*
Then the equivalence relation on X

$$\mathcal{R} \ \vee \ \{(y, \alpha(y))\,;\ y \in Y_0\}$$

generated by $\mathcal{R}$ and Graph (α) is orbit equivalent to $\mathcal{R}$ and therefore is affable.

8 Classification up to Orbit Equivalence of Minimal $\mathbb{Z}$-Actions

Let φ be a Cantor minimal system (i.e. a minimal action of $\mathbb{Z}$ on the Cantor set). The first step of the strategy outlined in section 6, namely to show that $\mathcal{R}_\varphi$ is affable is based on the following construction, that we sketch now.

Let $(U_n)_{n\geq 1}$ be a decreasing sequence of clopen subsets of X, whose intersection is a single point y. For $n \geq 1$, let $\mathcal{R}_n$ denote the equivalence relation on X generated by $\{(x, \varphi(x)) \mid x \in X \setminus U_n\}$. As φ is minimal and as the first return map of φ on U_n is continuous, we have that $\mathcal{R}_n$ is compact and open. As $(U_n)_{n\geq 1}$ forms a decreasing sequence of clopen sets, the sequence of the equivalence relations $(\mathcal{R}_n)_{n\geq 1}$ is increasing and their union $\mathcal{R}_y$ is an AF-relation. Every $\mathcal{R}_y$-class is also a φ-orbit, except for the orbit of the point y and $\mathcal{R}_\varphi = \mathcal{R}_y \ \vee \{(y, \varphi(y))\}$. With $Y_0 = \{y\}$, $Y_1 = \{\varphi(y)\}$ and $\alpha = \varphi$, we can apply Theorem 16 and we get:

Theorem 17. *([GPS1]). Let φ be a Cantor minimal system. Then the equivalence relation $\mathcal{R}_\varphi$ is orbit equivalent to an AF-relation.*

As a consequence of this result and of Theorem 14, we have:

Theorem 18. *Two Cantor minimal systems φ and ψ are orbit equivalent if and only if $(D_m(X,\mathcal{R}_\varphi), D_m(X,\mathcal{R}_\varphi)^+, [1])$ and $(D_m(X,\mathcal{R}_\psi), D_m(X,\mathcal{R}_\psi)^+, [1])$ are (order) isomorphic.*

Remark 19. a) Let φ be a Cantor minimal system and $C^*(X,\varphi)$ be the associated C^*-crossed product. If μ is any φ-invariant probability measure on X, recall that $C^*(X,\varphi)$ can be realized as the C^*-subalgebra of the bounded linear operator algebra $\mathcal{B}(L^2(X,\mu))$ generated by $C(X)$ acting as multiplication operators and the unitary operator $u = u_\varphi$, defined by

$$u\xi(x) = \xi(\varphi^{-1}x) \ , \text{ for } x \in X \text{ and } \ \xi \in L^2(X,\mu) \, .$$

If Y is a non-empty closed subset of X, let A_Y^φ be the C*-subalgebra of $C^*(X,\varphi)$ generated by $C(X)$ and $uC_0(X \setminus Y)$, where $C_0(X \setminus Y)$ denotes the continuous functions vanishing on Y.

Let $(U_n)_{n\geq 1}$ be a decreasing sequence of clopen subsets of X, whose intersection is a single point y. In [P], I. Putnam proved that

- A_y^φ is an AF algebra which is the closure of the increasing union of the finite dimensional algebras $A_{U_n}^\varphi$, for $n \geq 1$
- the inclusion map $i : A_y^\varphi \to C^*(X,\varphi)$ induces an isomorphism $i_* : K_0(A_y^\varphi) \to K_0(C^*(X,\varphi))$ of ordered groups preserving the distinguished order units (the class of the identity operator).
- $K_0(C^*(X,\varphi))$ is order isomorphic to $D(X,\mathcal{R}_\varphi)$ by a map preserving the order units.

Let $C^*(\mathcal{S})$ denote the reduced C*-algebra associated to an étale equivalence relation $\mathcal{S}$ (see [Pa] and [R] for example). If for $n \geq 1$, as introduced above, $\mathcal{R}_n$ denotes the equivalence relation generated by $\{(x,\varphi(x)) \mid x \in X \setminus U_n\}$ and $\mathcal{R}_y$ their union, then $C^*(\mathcal{R}_n)$ is isomorphic to $A_{U_n}^\varphi$ and A_y^φ to $C^*(\mathcal{R}_y)$.

If φ and ψ are two orbit-equivalent minimal homeomorphisms acting on the Cantor set X and if $F \in \text{Homeo}\,(X)$ is an orbit map between them, recall that the orbit cocycles m an n associated to F are the integer-valued functions on X defined for $x \in X$, by

$$F \circ \varphi(x) = \psi^{n(x)} \circ F(x) \quad \text{and} \quad F \circ \varphi^{m(x)}(x) = \psi \circ F(x) \, .$$

By a theorem of M. Boyle (see [GPS1], Thm 1.4), if one of the orbit cocycles is continuous, then φ and ψ are *flip-conjugate* (i.e., φ is conjugate to either ψ or ψ^{-1}).

Definition 20. *Let φ and ψ be two orbit-equivalent minimal homeomorphisms acting on the Cantor set X. Then φ and ψ are strong orbit equivalent if there exists an orbit map F so that the associated orbit cocycles $m, n : X \to \mathbb{Z}$ each have at most one point of discontinuity.*

Let us finish this section on the classification of Cantor minimal systems by stating the following two results:

Theorem 21. *Let φ and ψ be two minimal homeomorphisms acting on the Cantor set X. Then the two étale equivalence relations $\mathcal{R}_\varphi$ and $\mathcal{R}_\psi$ are isomorphic if and only if φ and ψ are flip-conjugate.*

Theorem 22. *Let φ and ψ be two minimal homeomorphisms acting on the Cantor set X. For any two points y_1 and y_2 of X, the AF-equivalence relations $\mathcal{R}_{\varphi,y_1}$ and $\mathcal{R}_{\psi,y_2}$ are isomorphic if and only if φ and ψ are strongly orbit equivalent.*

For the first result, if φ and ψ are flip-conjugate, then $\mathcal{R}_\varphi$ and $\mathcal{R}_\psi$ are clearly isomorphic. Conversely, if F is an orbit map such that $F \times F$ is a homeomorphism from $\mathcal{R}_\varphi$ to $\mathcal{R}_\psi$, then it follows that the orbit cocycles are bounded and by M. Boyle's theorem φ and ψ are flip-conjugate.

The second one follows from [GPS2], Lemma 4.13 and Corollary 1.3, in combination with Theorem 2.1 of [GPS1].

9 Classification up to Orbit Equivalence of Minimal $\mathbb{Z}^2$-Actions

Let φ be a minimal free action of $\mathbb{Z}^2$ on the Cantor set. To fulfill the second step of the strategy outlined in section 6, we use cocycles for the action to create the AF-relation and its extension. The drawback of this method is that it needs to assume the existence of sufficiently many cocycles with conditions of positivity and smallness we will define below. Results about the existence are still partial, although they do exist for several examples of interest.

In [F], Forrest (see also [Ph] for another treatment) produced large AF-subrelations of the orbit relation $\mathcal{R}_\varphi$. Such subrelations also appear implicitly in works of Bellissard, Benedetti and Gambaudo [BBG] and also in Benedetti and Gambaudo [BG]. But their methods do not keep track of the difference between the AF-subrelation and $\mathcal{R}_\varphi$ and therefore do not allow the use of the absorption theorem.

9.1 Cocycles and Positive Cocycles

Before stating our main results, we need to recall some basic notions about cocycles whose basic references are [FM, R].

Definition 23. *Let φ be a free action of $\mathbb{Z}^2$ on a compact space X.*

A $\mathbb{Z}$-valued one-cocycle for φ is a continuous function $\theta : X \times \mathbb{Z}^2 \to \mathbb{Z}$ such that , for all $x \in X$ and $m, n \in \mathbb{Z}^2$, we have

$$\theta(x, m+n) = \theta(x, m) + \theta(\varphi^m(x), n)\,.$$

If $f \in C(X, \mathbb{Z})$, then the function $bf(x,n) = f(\varphi^n(x)) - f(x)$ is called a coboundary.

The set $Z^1(X,\varphi)$ of all cocycles forms a group under addition, the set $B^1(X,\varphi)$ of all coboundaries a subgroup and we denote by $H^1(X,\varphi) = Z^1(X,\varphi)/B^1(X,\varphi)$ the first cohomology group. Using the equivalence relation induced by φ on X, cocycles and coboundaries can also be viewed as continuous homomorphisms from $\mathcal{R}_\varphi$ to $\mathbb{Z}$. Therefore if θ is a cocycle, its kernel $ker(\theta) = \{(x,y) \in \mathcal{R}_\varphi \,|\, \theta(x,y) = 0\,\}$ is a closed subequivalence of $\mathcal{R}_\varphi$.

We introduce now the notion of strict positivity for cocycles.

Definition 24. *Let φ be a free action of $\mathbb{Z}^2$ on a compact space X and let C be a subset of $\mathbb{Z}^2$. If θ is a cocycle, then it is*

1. *positive with respect to C if $\theta\,(X \times C) \geq 0$.*
2. *proper with respect to C if the map $\theta : X \times C \to \mathbb{Z}$ is proper (i.e., the pre-image of any finite set is compact).*
3. *strictly positive with respect to C if it is proper and positive with respect to C.*

Condition (2) of this definition is the key property we use to produce compact open subequivalence of $\mathcal{R}_\varphi$. Indeed we have:

Proposition 25. *([GPS3], Prop. 5.12) Let (X,φ) be as in 24, ξ and η be cocycles for (X,φ) and let $C, C' \subset \mathbb{Z}^2$. If ξ is proper on C and η is proper on C' and*

$$C \cup (-C) \cup C' \cup (-C') = \mathbb{Z}^2\ ,$$

then $ker(\xi) \cap ker(\eta) = \{(x,y) \in \mathcal{R}_\varphi \,|\, \xi(x,y) = \eta(x,y) = 0\,\}$ is a compact open subequivalence relation of $\mathcal{R}_\varphi$.

The sets C and C' we will need have the following special form:

Definition 26. *For $0 \leq r, r' \leq \infty$, we define*

$$C(r,r') = \{(i,j) \in \mathbb{Z}^2 \mid j \leq ri\,,\ j \leq r'i\,\}\,.$$

with the convention $0 \cdot \infty = 0$.

In addition to the notion of positive cocycle, we use the notion of small cocycle as follows.

Definition 27. *Let θ be a cocycle for (X,φ) and let M be a positive integer.*

Then θ is smaller or equal to M^{-1} if $|\theta(X,n)| \leq 1$ for all $x \in X$ and $n = (n_1, n_2) \in \mathbb{Z}^2$ with $||n||_\infty = max\,\{|n_1|, |n_2|\} \leq M$ and we say that θ is small if $\theta \leq \frac{1}{2}$.

If φ is a free minimal action of $\mathbb{Z}^2$ on a Cantor set X, finding small, positive cocycles reduces to finding clopen with the following properties.

Theorem 28. *Let φ be a free minimal action of $\mathbb{Z}^2$ on a Cantor set X. Let a, b be generators for $\mathbb{Z}^2$. Suppose that for any $N \geq 1$, there are clopen sets A and B such that*

1. $A \cap \varphi^{-a}(B) = \varphi^{-b}(A) \cap B = \emptyset$,
2. $A \cup \varphi^{-a}(B) = \varphi^{-b}(A) \cup B$,
3. the sets $\varphi^{i\,(a+b)}(A \cup \varphi^{-a}(B))$ *are disjoint for* $0 \leq i \leq N$.

Then for any $M \geq 1$*, there exists a cocycle* θ *which is strictly positive on* $C = \{ia + jb \mid i, j \geq 0\,\}$ *and* $\theta \leq M^{-1}$.

The preceding theorem is the result we use to find small strictly positive cocycles. for the following two classes of minimal free $\mathbb{Z}^2$-actions on the Cantor set.

Remark that any extension of a free minimal $\mathbb{Z}^2$-Cantor system with small strictly positive cocycles has the same property.

Example 29. Rotations of the group of p-adic integers.
Let p be a prime number and $X = \prod_{k=0}^{\infty} \mathbb{Z}/p\mathbb{Z}$ be the abelian group of p-adic integers. If α , β are two $\mathbb{Z}$-linearly independent elements of X such that either α_0 or β_0 is non-zero, then the action φ given for all $x \in X$ and $(i, j) \in \mathbb{Z}^2$ by

$$\varphi^{(i,j)}(x) = x - i\alpha - j\beta\,, \quad \text{for } x \in X \text{ and } \ (i, j) \in \mathbb{Z}^2\,,$$

is minimal and free.

As the subgroup of X generated by either α or β is dense, we can assume that one of the generator, α for example, is $(1, 0, 0, \cdots, 0, \cdots)$. Let $C(0, m)$ denote the cyclinder set $\{x \in X \mid x_0 = x_1 = \cdots = x_m = 0\}$ and $k(m)$ the smallest positive integer such that

$$\beta(C(0, m)) = \{x + \beta \in X \mid x_0 = \cdots = x_m = 0\} = \alpha^{k(m)}(C(0, m))\,.$$

Then the pair of clopen sets $A = C(0, m)$ and $B = \coprod_{l=0}^{k(m)} \alpha^l(C(0, m))$ satisfies the first two conditions of Theorem 28 and therefore defines a strictly positive cocycle. A much finer construction is necessary to get a small, strictly positive cocycle.

Notice that in this example there is a short exact sequence

$$0 \to \mathbb{Z} \to H^1(X, \varphi) \to \mathbb{Z}[1/p] \to 0\,.$$

Example 30. Rotations of a disconnected circle.
Let $0 < \alpha, \beta < \frac{1}{2}$ be two real numbers such that $\{1, \alpha, \beta\}$ is $\mathbb{Q}$-linearly independent and let us consider the action of $\mathbb{Z}^2$-action on the circle $\mathbb{R}/\mathbb{Z}$, by rotating by α and β. We then disconnect the circle along an orbit replacing each point by two separated ones and obtain a copy of the Cantor set. More precisely, if $Cut \subset \mathbb{R}$ denotes the subgroup $\{k + n\alpha + m\beta \mid k, n, m \in \mathbb{Z}\,\}$, we define a linear order on $\tilde{X} = \mathbb{R} \cup \{a' \mid a \in Cut\,\}$ by setting $a' < b$, $a < b'$, $a' < b'$ if $a < b$ and $a < a'$ for all $a \in \mathbb{R}$ and consider the order topology on $\tilde{X}$. The action by translation of $\mathbb{Z} + \alpha\,\mathbb{Z} + \beta\,\mathbb{Z}$ on $\tilde{X}$ induces then a (minimal, free) action φ of $\alpha\,\mathbb{Z} + \beta\,\mathbb{Z}$ on the Cantor set $X = \tilde{X}/\mathbb{Z}$.

For any pair of generators $a, b \in \alpha\mathbb{Z} + \beta\mathbb{Z}$, the homeomorphisms φ^a and φ^b are again rotations of the cut-up circle X. Let N be a positive integer. By minimality of the rotation $R_a \times R_b$ on $\mathbb{R}^2/\mathbb{Z}^2$, there exist a positive integer q and integers i and j such that

$$0 < qa - i < \frac{1}{2N}\ , \quad 0 < qb - j < \frac{1}{2N}\ .$$

Then for $0 \leq m, n < N$ and $k \in \mathbb{Z}$, we have

$$\frac{k}{q} \leq \frac{k - im - jn}{q} + ma + nb \leq \frac{k+1}{q}\ .$$

For each $0 \leq k < q$, we can approximate $\frac{k}{q}$ by an element $x_k \in Cut$ such that for $0 \leq m, n < N$,

$$x_k < x_{k-im-jn} + ma + nb < x_{k+1}\ .$$

Then the pair of clopen sets of X

$$A = \cup_{k=0}^{q-1}[x_k, x_{k-i} + a)\ , \quad B = \cup_{k=0}^{q-1}[x_k, x_{k-j} + b)$$

with $k - i$ and $k - j$ interpreted modulo q, satisfy the assumptions of Theorem 28 and therefore defines a small stictly positive cocycle.

Notice that the first cohomology group of this example was computed by Forrest and Hunton in [FH] and is equal to $\mathbb{Z}^3$.

Remark 31. If φ is a minimal $\mathbb{Z}^2$-action on the Cantor set, its first cohomology group $H^1(X, \varphi)$ always contains $\mathbb{Z}^2$ as a subgroup. There exists an example of such an action such that $H^1(X, \varphi) = \mathbb{Z}^2$. But this action is not free. It is not known if the first cohomology group of a free, minimal $\mathbb{Z}^2$-action on the Cantor set has always $\mathbb{Z}^2$ as a proper subgroup.

9.2 The Main Results of [GPS3]

Our main result, Theorem 32, whose proof is very long, states that if a free, minimal action φ of $\mathbb{Z}^2$ on the Cantor set possesses arbitrary small, strictly positive cocycles for sufficiently many cones, then the induced étale equivalence relation $\mathcal{R}_\varphi$ is affable.

Theorem 32. *Let (X, φ) be a free, minimal action of $\mathbb{Z}^2$ on the Cantor set. Suppose that there are positive numbers r_∞, s_∞ with $s_\infty^{-1} - r_\infty^{-1} \geq 1$ satisfying the following: For every $\varepsilon > 0$, there are positive real numbers $r_\infty + \varepsilon > r > r' > r_\infty$ so that for every $M \geq 1$, there is a cocycle θ on (X, φ) such that*

$$1.\quad \theta \text{ is strictly positive on } C(r, r'), \quad \text{and} \ 2.\quad \theta \leq M^{-1}\ .$$

Similarly, for every $\varepsilon > 0$, there are positive real numbers $s_\infty - \varepsilon < r < r' < s_\infty$ such that for every $M \geq 1$, there is a cocycle θ on (X, φ) satisfying conditions 1 and 2.

Then the étale equivalence relation $\mathcal{R}_\varphi$ is affable.

The examples 29 and 30 described above satisfy the following stronger hypotheses.

Corollary 33. *Let (X, φ) be a free, minimal action of $\mathbb{Z}^2$ on the Cantor set. Suppose that for every $a, b \in \mathbb{Z}^2$ which generates $\mathbb{Z}^2$ as a group and for every $M \geq 1$, there is a cocycle θ on (X, φ) such that*

1. *θ is strictly positive on $\{ia + jb \mid i, j \geq 0\}$, and* 2. *$\theta \leq M^{-1}$.*

Then the étale equivalence relation $\mathcal{R}_\varphi$ is affable.

As a consequence of Theorems 17, 32, and 14, we then get:

Theorem 34. *For $i = 1, 2$, let $(X_i, \mathcal{R}_i)$ be étale equivalence relations where, for each i, X_i is totally disconnected and $(X_i, \mathcal{R}_i)$ is minimal and one of the following conditions are satisfied:*

1. *$\mathcal{R}_i$ is an AF-relation,*
2. *$\mathcal{R}_i$ arises from a free action of $\mathbb{Z}$, or*
3. *$\mathcal{R}_i$ arises from a free action of $\mathbb{Z}^2$ satisfying the hypotheses of 32.*

Then the two equivalence relations are orbit equivalent if and only if there is an order isomorphism from $D_m(X_1, \mathcal{R}_1)$ to $D_m(X_2, \mathcal{R}_2)$ preserving the distinguished order units.

10 Further Developments

Let (X, φ) be a Cantor minimal $\mathbb{Z}^2$-system conjugated to the product of two Cantor minimal $\mathbb{Z}$-systems (X_1, φ_1) and (X_2, φ_2). By Theorem 17, we have that $\mathcal{R}_{\varphi_1}$ and $\mathcal{R}_{\varphi_2}$ are orbit equivalent to two AF-relations $\mathcal{R}_1$ and $\mathcal{R}_2$. As the product of two AF-equivalence relations is also AF, we have that $\mathcal{R}_\varphi$ is also affable.

If (Y, ψ) is an extension of the product Cantor system $(X_1 \times X_2, \varphi_1 \times \varphi_2)$, it is not necessarily a product. Therefore the above argument cannot be used to show the affability of (Y, ψ). In [M1], H. Matui constructs an AF-subequivalence of $\mathcal{R}_\psi$ satisfying the assumptions of the absorption theorem to prove the following:

Theorem 35. *([M1]). Let $\pi : (Y, \psi) \to (X, \varphi)$ be an extension between Cantor minimal $\mathbb{Z}^2$-systems. If (X, φ) is conjugate to a product of two Cantor minimal $\mathbb{Z}$-systems, then $\mathcal{R}_\psi$ is affable.*

Recall (see for example [KP], p. 180) that a tiling T in $\mathbb{R}^2$ gives rise to an action of $\mathbb{R}^2$ on its continuous hull Ω_T. If T has finite local complexity and is strongly aperiodic, the hull Ω_T is compact and does not contain any periodic tilings. If moreover T is repetitive, then the dynamical system $(\Omega_T, \mathbb{R}^2)$ is minimal.

For each tile type (or labeled tile type) t in T, let us choose a point $x(t)$, called a puncture, in the interior of t. Now each tile $t \in T$ is given a puncture $x(t)$ such that if t_1 and t_2 are two tiles with $t_2 = t_1 + x$ for some $x \in \mathbb{R}^2$, then $x(t_2) = x(t_1) + x$.

The set of all the tilings $T' \in \Omega_T$ such that the origin is a puncture of some tiles t in T' is called the discrete hull Ω_{punc} of T. With the above conditions on the tiling, Ω_{punc} is a Cantor set and is a transversal to the $\mathbb{R}^2$-action.

An equivalence relation $\mathcal{R}_{punc}$ is defined on Ω_{punc} as follows:

$$\mathcal{R}_{punc} = \{(T_1, T_2) \,|\, T_i \in \Omega_{punc} \text{ and } \exists\, x \in \mathbb{R}^2 \,:\, T_1 = T_2 + x\,\}\,.$$

Then $\mathcal{R}_{punc}$ is the restriction to Ω_{punc} of the equivalence relation induced by the $\mathbb{R}^2$-action on the continuous hull Ω_T.

Provided with the following topology: a sequence $(T_n, T_n + x_n)$ in $\mathcal{R}_{punc}$ converges to $(T, T + x)$ if and only if $T_n \to T$ and $x_n \to x$, $\mathcal{R}_{punc}$ is an étale equivalence relation.

In [M2], H. Matui studies the equivalence relation $\mathcal{R}_{punc}$ associated to a substitution tiling. Recall that a substitution tiling system in $\mathbb{R}^2$ consists of a pair $(\mathcal{V}, \omega)$ where $\mathcal{V}$ is a finite collection of polygons in $\mathbb{R}^2$, the prototiles, and ω is a substitution rule. We also have an inflation constant $\lambda > 1$ such that for every $p \in \mathcal{V}$, $w(p)$ is a finite collection of tiles (a tile is a translate of one prototile) with pairwise disjoint interiors and their union is $\lambda p = \{\lambda v \mid v \in p\}$. The Penrose tiling is an example of a substitution tiling.

For a substitution tiling system which is primitive, aperiodic and satisfies the finite pattern condition, I. Putnam constructs in [P1] a minimal AF subequivalence relation $\mathcal{R}$ of $\mathcal{R}_{punc}$. The equivalence relation $\mathcal{R}$ is too large to apply the Absorption Theorem 16. In [M2], Matui constructs a smaller AF-subequivalence relation $\mathcal{R}' \subset \mathcal{R}$ satisfying the conditions of Theorem 16 and obtains:

Theorem 36. *([M2]). Let $(\mathcal{V}, \omega)$ be a substitution tiling system in $\mathbb{R}^2$ as above. Then the equivalence relation $\mathcal{R}_{punc}$ on Ω_{punc} is affable.*

In a work in progress, Giordano, Matui, Putnam and Skau have generalized the Absorption Theorem presented in Section 7. With this gneralization, the first step of the strategy presented in Section 6 is now easier to implement. In particular for a minimal free action of $\mathbb{Z}^2$ on the Cantor set, we are now able to construct a minimal AF-subequivalence relation of $\mathcal{R}_\varphi$ satisfying the assumptions of the new absorption theorem without having to use cocycles. Theorem 34 can therefore be extended to cover all free minimal action of $\mathbb{Z}^2$ on the Cantor set.

References

[BBG] J. Bellissard, R. Benedetti and J.-M. Gambaudo, *Spaces of Tilings, Finite Telescopic Approximations and Gap-Labelling*, preprint.

[BG] R. Benedetti and J.-M. Gambaudo, *On the dynamics of G-solenoids: application to Delone sets*, Erg. Th. and Dyn. Sys.,**23**, (2003), p. 673–691.

[CFW] A. Connes, J. Feldman and B. Weiss, *An amenable equivalence relation is generated by a simple transformation.* J. Erg. Th. and Dyn. Sys.,**1**, (1981), p. 431–450.

[DJK] R. Dougherty, S. Jackson and A.S. Kechris, *The structure of hyperfinite Borel equivalence relations*, Trans. Amer. Math. Soc., **341**, (1994), p. 193–225.

[D1] H. Dye, *On groups of measure preserving transformations I*, American J. Math, **81**, (1959), 119–159.

[D2] H. Dye, *On groups of measure preserving transformations II*, American J. Math, **85**, (1963), 551–576.

[Ef] E.G. Effros, *Dimensions and C*-algebras*, Conf. Board Math. Sci., **46**, Amer. Math. Soc., Providence, R.I., 1981.

[FM] J. Feldman and C.C. Moore, *Ergodic equivalence relations, cohomology and von Neumann algebras I*, Trans. Amer. Math. Soc., **234**, (1977), p. 289-324.

[F] A. Forrest, *A Bratteli diagram for commuting homeomorphism of the Cantor set*, Internat. J. Math. **11** (2000), p. 177–200.

[FH] A. Forrest and J. Hunton *The cohomology and K-theory of commuting homeomorphisms of the Cantor set.* Erg. Th. and Dyn. Sys.,**19**,(1999), 611-625.

[GPS1] T. Giordano, I.F. Putnam and C.F. Skau, *Topological orbit equivalence and C*-crossed products*, J. reine angew. Math. **469** (1995), p. 51–111.

[GPS2] T. Giordano, I.F. Putnam and C.F. Skau, *Affable equivalence relations and Cantor minimal*, Ergod. Theor. and Dynam. Syst., **23**, (2004), p. 441–475.

[GPS3] T. Giordano, I.F. Putnam and C.F. Skau, *The Orbit Structure of Cantor Minimal $\mathbb{Z}^2$-systems*, preprint.

[HPS] R.H. Herman, I.F. Putnam and C.F. Skau, *Ordered Bratteli diagrams, dimension groups, and topological dynamics*, Internat. J. Math. **3** (1992), p. 827–864.

[HM] G. Hjorth and M. Molberg, *Free continuous actions on zero dimensional spaces*, to appear in Topology and its applications.

[KP] J. Kellendonk and I.F. Putnam, *Tilings, C*-algebras, and K-theory*, Directions in Mathematical Quasicrystals, 117–206, CRM Monogr. Ser. **13**, Amer. Math. Soc., Providence, RI, 2000.

[Kr] W. Krieger, *On a dimension for a class of homeomorphism groups*, Math. Ann., **252**, (1980), p. 87–95.

[K] K. Kuratowski, *Topology*, Vol II, Academic Press, New-York 1968.

[M1] H. Matui, *A short proof of affability for certain Cantor minimal $\mathbb{Z}^2$-systems*, preprint.

[M2] H. Matui, *Affability of equivalence relations arising from two-dimensional substitution tilings*, preprint.

[OW] D.S. Ornstein and B. Weiss, *Ergodic theory of amenable group actions I: The Rohlin lemma*, Bull. Amer. Math. Soc., **2**, (1980), p. 161–164.

[Pa] A. L. T. Paterson, *Groupoids, Inverse Semigroups and their Operator Algebras.* (Progress in Mathematics, vol. 170), Birkhäuser, Boston-Basel-Berlin, 1999.

[Ph] N.C. Phillips, *Crossed product of the Cantor set by a free, minimal action of $\mathbb{Z}^n$*, Comm. Math. Phys., to appear.

[P] I.F. Putnam, *The C*-algebras associated with minimal homeomorphisms of the Cantor set*, Pacific J. Math.., **136**, (1989), p. 329–353.

[P1] I.F. Putnam, *The ordered K-theory of C*-algebras associated with substitution tilings*, Comm. Math. Phys., **214**, (2000), p. 593–605.

[R] J. Renault, *A groupoid approach to C^* -algebras.* Lect. Notes Math. **739**, Springer-Verlag, 1980.

[R1] J. Renault, *AF-equivalence relations and their cocycles.* Preprint (2001).

[W] B. Weiss, Private communications.

Outer Actions of a Group on a Factor

Yoshikazu Katayama[1] and Masamichi Takesaki[2]

[1] Department of Mathematics, Osaka Kyoiku University, Osaka, Japan
[2] Department of Mathematics, UCLA and Department of Mathematical Sciences, Norwegian University of Science and Technology

First, I will discuss the characteristic square of a factor $\mathcal{M}$:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathcal{U}(\mathcal{C}) & \xrightarrow{\partial_\theta} & \mathrm{B}^1_\theta(\mathbb{R},\mathcal{U}(\mathcal{C})) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{U}(\mathcal{M}) & \longrightarrow & \tilde{\mathcal{U}}(\mathcal{M}) & \xrightarrow{\partial_\theta} & \mathrm{Z}^1_\theta(\mathbb{R},\mathcal{U}(\mathcal{C})) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \mathrm{Ad}} & & \Big\downarrow{\scriptstyle \widetilde{\mathrm{Ad}}} & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Int}(\mathcal{M}) & \longrightarrow & \mathrm{Cnt_r}(\mathcal{M}) & \xrightarrow{\dot{\partial}_\theta} & \mathrm{H}^1_\theta(\mathbb{R},\mathcal{U}(\mathcal{C})) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

where $\{\mathcal{C}, \mathbb{R}, \theta\}$ is the flow of weights on $\mathcal{M}$; $\widetilde{\mathcal{U}}(\mathcal{M})$ is the extended unitary group of $\mathcal{M}$, i.e., the normalizer of $\mathcal{M}$ in the unitary group $\mathcal{U}(\widetilde{\mathcal{M}})$ of the core $\widetilde{\mathcal{M}}$ of $\mathcal{M}$. The core $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ is the von Neumann algebra generated by the imaginary power $\{\varphi^{it} : t \in \mathbb{R}, \varphi \in \mathfrak{W}_0(\mathcal{M})\}$ of faithful semi-finite normal weights on $\mathcal{M}$. Scaling $\varphi \mapsto e^{-s}\varphi, s \in \mathbb{R}$, gives rise to the one parameter automorphism group $\{\theta_s : s \in \mathbb{R}\}$ of $\widetilde{\mathcal{M}}$ such that

$$\mathcal{M} = \widetilde{\mathcal{M}}^{\theta} \quad \text{and} \quad \mathcal{M}' \cap \widetilde{\mathcal{M}} = \mathcal{C}.$$

The normalizer $\widetilde{\mathcal{U}}(\mathcal{M})$ of $\mathcal{M}$ in the unitary group $\mathcal{U}(\widetilde{\mathcal{M}})$ of $\widetilde{\mathcal{M}}$ gives the extended modular automorphism group $\mathrm{Cnt_r}(\mathcal{M})$ as every $u \in \widetilde{\mathcal{U}}(\mathcal{M})$ gives an automorphism $\widetilde{\mathrm{Ad}}(u)(x) = uxu^*, x \in \mathcal{M}$.

Looking at the middle vertical exact sequence:

$$1 \longrightarrow \mathcal{U}(\mathcal{C}) \longrightarrow \tilde{\mathcal{U}}(\mathcal{M}) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{Cnt_r}(\mathcal{M}) \longrightarrow 1$$

choose a cross-section: $\alpha \in \mathrm{Cnt_r}(\mathcal{M}) \mapsto u(\alpha) \in \widetilde{\mathcal{U}}(\mathcal{M})$ such that $\alpha = \widetilde{\mathrm{Ad}}(u)(\alpha))$. Then we have:

$$\mu(\alpha, \beta) = u(\alpha)u(\beta)u(\alpha\beta)^* \in \mathcal{U}(\mathcal{C});$$
$$\lambda(\alpha, \gamma) = \gamma(u(\gamma^{-1}\alpha\gamma))u(\alpha)^* \in \mathcal{U}(\mathcal{C}), \quad \alpha, \beta \in \mathrm{Cnt_r}(\mathcal{M}), \gamma \in \mathrm{Aut}(\mathcal{M}).$$

The pair (λ, μ) is a characteristic cocycle of V.F.R. Jones and gives rise to the characteristic invariant $\Theta(\mathcal{M})$ in the relative cohomology group $\Lambda(\mathrm{Aut}(\mathcal{M}) \times \mathbb{R}, \mathrm{Cnt_r}(\mathcal{M}), \mathcal{U}(\mathcal{C}))$, which was named the intrinsic invariant of $\mathcal{M}$ in [4].

If $\alpha : g \in G \mapsto \alpha_g \in \mathrm{Aut}(\mathcal{M})$ is an action of a group G, then the pullback $\chi(\alpha) = \alpha^*(\Theta(\mathcal{M})) \in \Lambda_{\mathrm{mod}(\alpha)\times\theta}(G \times \mathbb{R}, N, \mathcal{U}(\mathcal{C}))$ with $N = \alpha^{-1}(\mathrm{Cnt_r}(\mathcal{M}))$ is a cocycle conjugacy invariant. In the case that $\mathcal{M}$ is an approximately finite dimensional factor and G is a countable discrete amenable group, then the triplet $\{\mathrm{mod}(\alpha), \alpha^{-1}(\mathrm{Cnt_r}(\mathcal{M})), \chi(\alpha)\}$ form a complete invariant of the cocycle conjugacy class of α.

To move on one step further to outer actions, we first make the definition.

Definition 1. *A map* $\alpha\colon g \in G \mapsto \alpha_g \in \mathrm{Aut}(\mathcal{M})$ *is called an* outer action *if*

$$\alpha_g \circ \alpha_h \equiv \alpha_{gh} \mod \mathrm{Int}(\mathcal{M}), \quad g, h \in G.$$

We usually assume that $\alpha_e = \mathrm{id}$ *for the identity* $e \in G$. *If*

$$\alpha_g \notin \mathrm{Int}(\mathcal{M}), \quad g \neq e,$$

then it is called a free *outer action.*

Remark 2. One should not confuse this with the concept of *free actions*.

Consider the quotient group $\mathrm{Out}(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})/\mathrm{Int}(\mathcal{M})$ and fix a cross-section: $g \in \mathrm{Out}(\mathcal{M}) \mapsto \alpha_g \in \mathrm{Aut}(\mathcal{M})$ of the quotient map $\pi : \alpha \in \mathrm{Aut}(\mathcal{M}) \mapsto [\alpha] \in \mathrm{Out}(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})/\mathrm{Int}(\mathcal{M})$ and also choose a Borel cross-section $\alpha \in \mathrm{Cnt_r}(\mathcal{M}) \mapsto u(\alpha) \in \widetilde{\mathcal{U}}(\mathcal{M})$ in such a way that $u(\alpha) \in \mathcal{U}(\mathcal{M})$ for every $\alpha \in \mathrm{Int}(\mathcal{M})$. Then we have for $g, h, k \in \mathrm{Out}(\mathcal{M})$

$$u(g, h) = u(\alpha_g \circ \alpha_h \circ \alpha_{gh}^{-1}) \in \mathcal{U}(\mathcal{M}), ;$$
$$c(g, h, k) = \alpha_g(u(h, k))u(g, hk)\{u(g, h)u(gh, k)\}^* \in \mathbb{T}.$$

The three variable function c is indeed a cocycle $c \in \mathrm{Z}^3(\mathrm{Out}(\mathcal{M}), \mathbb{T})$. The cohomology class $[c] \in \mathrm{H}^3(\mathrm{Out}(\mathcal{M}), \mathbb{T})$ is called the *intrinsic obstruction* and denoted by $\mathrm{Ob}(\mathcal{M})$. If α is an outer action of G on $\mathcal{M}$, then the pull back $\mathrm{Ob}(\alpha) = \alpha^*(\mathrm{Ob}(\mathcal{M}))$ is an invariant of the outer conjugacy class of α. If $\mathcal{M}$ is a factor of type II_1, then one can work directly on the obstruction, employing the Brower group trick. But in the case of type III, this direct method does not work. For example, the group $\mathrm{Cnt_r}(\mathcal{M})$ is not stable under the tensor product, while $\mathrm{Int}(\mathcal{M})$ is stable. To deal with this problem, we will do the following:

To each factor $\mathcal{M}$, we associate an invariant $\mathrm{Ob_m}(\mathcal{M})$ to be called the *intrinsic modular obstruction* as a cohomological invariant which lives in the "third" cohomology group:

$$\mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(\mathrm{Out}(\mathcal{M}) \times \mathbb{R}, \mathrm{H}^1_\theta(\mathbb{R}, \mathcal{U}(\mathcal{C})), \mathcal{U}(\mathcal{C}))$$

where $\{\mathcal{C}, \mathbb{R}, \theta\}$ is the flow of weights on $\mathcal{M}$. If α is an outer action of a countable discrete group G on $\mathcal{M}$, then the triple consisting of its modulus $\mathrm{mod}(\alpha) \in \mathrm{Hom}(G, \mathrm{Aut}_\theta(\mathcal{C}))$ together with $N = \alpha^{-1}(\mathrm{Cnt_r}(M))$ and the pull back

$$\mathrm{Ob_m}(\alpha) = \alpha^*(\mathrm{Ob_m}(\mathcal{M})) \in \mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G \times \mathbb{R}, N, \mathcal{U}(\mathcal{C})))$$

is called the *modular obstruction* of α and is an invariant of the outer conjugacy class of the outer action α.

We have proved that if the factor $\mathcal{M}$ is approximately finite dimensional and G is amenable, then this invariant uniquely determines the outer conjugacy class of α, and then every value of the triple occurs as the invariant of an outer action α of G on $\mathcal{M}$. In the case that M is a factor of type III_λ, $0 < \lambda \leq 1$, the modular obstruction group $\mathrm{H}^{\mathrm{out}}_{\alpha,\mathfrak{s}}(G \times \mathbb{R}, N, \mathcal{U}(\mathcal{C}))$ and the modular obstruction $\mathrm{Ob_m}(\alpha)$ take simpler forms. But this does not mean that our work is easier. The difficulties in this case can be seen in the fact that $\mathrm{Aut}(\mathcal{M})$ does not act on the discrete core, a fact that is overlooked sometimes.

References

1. A. Connes. Periodic automorphisms of the hyperfinite factor of type II_1. *Acta Math. Szeged*, 39:39–66, 1977.
2. A.J. Falcone and M. Takesaki. Non-commutative flow of weights on a von *Neu*mann algebra. *J. Functional Analysis*, 182:170–206, 2001.
3. V.F.R. Jones. Actions of finite groups on the hyperfinite type III factor. *Amer. Math. Soc. Memoirs*, 237, 1980.
4. Y. Katayama, C. Sutherland and M. Takesaki. The characteristic square of a factor and the cocycle conjugacy of discrete amenable group actions on factors. *Invent. Math.*, 132:331–380, 1998.
5. Y. Katayama and M. Takesaki. Outer actions of a countable discrete amenable group on approximately finite dimensional factors I, General theory. *Contemporary Mathematics, AMS von Neumann–Stone Memorial Issue*, 365:181–237, 2004.
6. M. Takesaki. *Theory of Operator Algebras I.* Springer-Verlag, 1979. Heidelberg, New York, Hong Kong, Tokyo.
7. M. Takesaki. *Theory of Operator Algebras II.* Springer-Verlag, 2002. Heidelberg, New York, Hong Kong, Tokyo.
8. M. Takesaki. *Theory of Operator Algebras III.* Springer-Verlag, 2002. Heidelberg, New York, Hong Kong, Tokyo.

Non-Separable AF-Algebras

Takeshi Katsura

Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, 060-0810, JAPAN katsura@math.sci.hokudai.ac.jp

Summary. We give two pathological phenomena for non-separable AF-algebras which do not occur for separable AF-algebras. One is that non-separable AF-algebras are not determined by their Bratteli diagrams, and the other is that there exists a non-separable AF-algebra which is prime but not primitive.

1 Introduction

In this paper, an AF-algebra means a C^*-algebra which is an inductive limit of finite dimensional C^*-algebras on *any* directed set. Equivalently,

Definition 1. *A C^*-algebra A is called an* AF-algebra *if it has a directed family of finite dimensional C^*-subalgebras whose union is dense in A.*

When an AF-algebra A is separable, we can find an increasing *sequence* of finite dimensional C^*-subalgebras whose union is dense in A. Thus for separable C^*-algebras, the above definition coincides with the one in many literatures (for example, [3]). For separable C^*-algebras, there exists one more equivalent definition of AF-algebras:

Proposition 2 (Theorem 2.2 of [1]). *A separable C^*-algebra A is an AF-algebra if and only if it is a locally finite dimensional C^*-algebra, which means that for any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$, we can find a finite dimensional C^*-subalgebra B of A such that* $\operatorname{dist}(x, B) < \varepsilon$ *for all $x \in \mathcal{F}$.*

To the best of the author's knowledge, it is still open that the above lemma is valid in general.

For each positive integer $n \in \mathbb{Z}_+$, $\mathbb{M}_n$ denotes the C^*-algebra of all $n \times n$ matrices. Any finite dimensional C^*-algebra A is isomorphic to $\bigoplus_{i=1}^{k} \mathbb{M}_{n_i}$ for some $k \in \mathbb{Z}_+$ and ${}^t(n_1, \ldots, n_k) \in \mathbb{Z}_+^k$. Let $B \cong \bigoplus_{j=1}^{k'} \mathbb{M}_{n'_j}$ be another finite dimensional C^*-algebra. A $*$-homomorphism $\varphi\colon A \to B$ is determined up to unitary equivalence by the $k' \times k$ matrix N whose (j,i)-entry is the multiplicity of the composition of the restriction of φ to $\mathbb{M}_{n_i} \subset A_\lambda$ and the natural surjection from B to $\mathbb{M}_{n'_j}$.

Definition 3. *Let Λ be a directed set with an order $\preceq$. An* inductive system of finite dimensional C^*-algebras $(A_\lambda, \varphi_{\mu,\lambda})$ *over Λ consists of a finite dimensional C^*-algebra A_λ for each $\lambda \in \Lambda$, and a $*$-homomorphism $\varphi_{\mu,\lambda}\colon A_\lambda \to A_\mu$ for each $\lambda, \mu \in \Lambda$ with $\lambda \prec \mu$ such that $\varphi_{\nu,\mu} \circ \varphi_{\mu,\lambda} = \varphi_{\nu,\lambda}$ for $\lambda \prec \mu \prec \nu$.*

A Bratteli diagram *of $(A_\lambda, \varphi_{\mu,\lambda})$ is the system $(n_\lambda, N_{\mu,\lambda})$ where $n_\lambda = {}^t((n_\lambda)_1, \ldots, (n_\lambda)_{k_\lambda}) \in \mathbb{Z}_+^{k_\lambda}$ satisfies $A_\lambda \cong \bigoplus_{i=1}^{k_\lambda} \mathbb{M}_{(n_\lambda)_i}$ and $N_{\mu,\lambda}$ is $k_\mu \times k_\lambda$ matrix which indicates the multiplicities of the restrictions of $\varphi_{\mu,\lambda}$ as above.*

A Bratteli diagram $(n_\lambda, N_{\mu,\lambda})$ satisfies $N_{\mu,\lambda} n_\lambda \leq n_\mu$ for $\lambda \prec \mu$, and $N_{\nu,\mu} N_{\mu,\lambda} = N_{\nu,\lambda}$ for $\lambda \prec \mu \prec \nu$. It is not difficult to see that when the directed set Λ is $\mathbb{Z}_+$, any system $(n_\lambda, N_{\mu,\lambda})$ satisfying these two conditions can be realized as a Bratteli diagram of some inductive system of finite dimensional C^*-algebras (see 1.8 of [1]). This does not hold for general directed set:

Example 4. Let $\Lambda = \{a, b, c, d, e\}$ with an order $a \succ b, c \succ d, e$. Let us define

$$n_a = (24), \quad n_b = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \quad n_c = \begin{pmatrix} 6 \\ 6 \end{pmatrix}, \quad n_d = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad n_e = \begin{pmatrix} 2 \\ 2 \end{pmatrix},$$

and

$$N_{a,b} = \begin{pmatrix} 3 & 3 \end{pmatrix}, \qquad N_{b,d} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad N_{b,e} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad N_{a,d} = \begin{pmatrix} 6 & 6 \end{pmatrix},$$
$$N_{a,c} = \begin{pmatrix} 2 & 2 \end{pmatrix}, \qquad N_{c,d} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}, \qquad N_{c,e} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \qquad N_{a,e} = \begin{pmatrix} 6 & 6 \end{pmatrix}.$$

These matrices satisfy $N_{\mu,\lambda} n_\lambda = n_\mu$ for $\lambda, \mu \in \Lambda$ with $\mu \succ \lambda$, and

$$N_{a,b} N_{b,d} = N_{a,c} N_{c,d} = N_{a,d}, \qquad N_{a,b} N_{b,e} = N_{a,c} N_{c,e} = N_{a,e}.$$

Thus the system $(n_\lambda, N_{\mu,\lambda})$ satisfies the two conditions above. However, one can see that this diagram never be a Bratteli diagram of inductive systems of finite dimensional C^*-algebras.

In 1.8 of [1], O. Bratteli showed that when the directed set Λ is $\mathbb{Z}_+$, a Bratteli diagram of an inductive system of finite dimensional C^*-algebras determines the inductive limit up to isomorphism. This is no longer true for general directed set Λ as the following easy example shows.

Example 5. Let X be an infinite set, and Λ be the directed set consisting of all finite subsets of X with inclusion as an order. We consider the following two inductive systems of finite dimensional C^*-algebras.

For each $\lambda \in \Lambda$, we define a C^*-algebra $A_\lambda = \mathcal{K}(\ell^2(\lambda)) \cong \mathbb{M}_{|\lambda|}$ whose matrix unit is given by $\{e_{x,y}\}_{x,y\in\lambda}$. For $\lambda, \mu \in \Lambda$ with $\lambda \subset \mu$, we define a $*$-homomorphism $\varphi_{\mu,\lambda} \colon A_\lambda \to A_\mu$ by $\varphi_{\mu,\lambda}(e_{x,y}) = e_{x,y}$. It is clear to see that this defines an inductive system of finite dimensional C^*-algebras, and the inductive limit is $\mathcal{K}(\ell^2(X))$.

For each $\lambda \in \Lambda$ with $n = |\lambda|$, we set $A'_\lambda = \mathbb{M}_n$ whose matrix unit is given by $\{e_{k,l}\}_{1\leq k,l\leq n}$. For $\lambda, \mu \in \Lambda$ with $\lambda \subset \mu$, we define a $*$-homomorphism $\varphi'_{\mu,\lambda} \colon A'_\lambda \to A'_\mu$ by $\varphi'_{\mu,\lambda}(e_{k,l}) = e_{k,l}$. It is clear to see that this defines an inductive system of finite dimensional C^*-algebras, and the inductive limit is $\mathcal{K}(\ell^2(\mathbb{Z}_+))$.

The above two inductive systems give isomorphic Bratteli diagrams, but the AF-algebras $\mathcal{K}(\ell^2(X))$ and $\mathcal{K}(\ell^2(\mathbb{Z}_+))$ determined by the two inductive systems are isomorphic only when X is countable.

In a similar way, we can find two inductive systems of finite dimensional C^*-algebras whose Bratteli diagrams are isomorphic, but the inductive limits are $\bigotimes_{x\in X} \mathbb{M}_2$ and $\bigotimes_{k=1}^{\infty} \mathbb{M}_2$ which are not isomorphic when X is uncountable.

By Example 5, we can see that G. A. Elliott's celebrated theorem of classifying (separable) AF-algebras using K_0-groups (Theorem 6.4 of [3]) does not follow for non-separable AF-algebras, because K_0-groups are determined by Bratteli diagrams. Example 5 is not so interesting because the inductive system $(A'_\lambda, \varphi'_{\mu,\lambda})$ has many redundancies and does not come from directed families of finite dimensional C^*-subalgebras. More interestingly, we can get the following whose proof can be found in the next section:

Theorem 6. *There exist two non-isomorphic AF-algebras A and B such that they have directed families of finite dimensional C^*-subalgebras which define isomorphic Bratteli diagrams.*

The author could not find such an example in which every finite dimensional C^*-subalgebras are isomorphic to full matrix algebras $\mathbb{M}_n$ (cf. Problem 8.1 of [2]).

As another pathological fact on non-separable AF-algebras, we prove the next theorem in Section 3.

Theorem 7. *There exists a non-separable AF-algebra which is prime but not primitive.*

It had been a long standing problem whether there exists a C^*-algebra which is prime but not primitive, until N. Weaver found such a C^*-algebra in [6]. Note that such a C^*-algebra cannot be separable.

Acknowledgments. The author is grateful to the organizers of the Abel Symposium 2004 for giving him opportunities to talk in the conference and to contribute in this volume. He is also grateful to George A. Elliott and Akitaka Kishimoto for useful comments. This work was partially supported by Research Fellowship for Young Scientists of the Japan Society for the Promotion of Science.

2 Proof of Theorem 6

In this section, we will prove Theorem 6. Let X be an infinite set, and Z be the set of all subsets z of X with $|z| = 2$.

For each $z \in Z$, we define a C^*-algebra M_z by $M_z = \mathbb{M}_2$. Elements of the direct product $\prod_{z\in Z} M_z$ will be considered as norm bounded functions f on Z such that $f(z) \in M_z$ for $z \in Z$. For each $z \in Z$, we consider $M_z \subset \prod_{z\in Z} M_z$ as a direct summand. We denote by $\bigoplus_{z\in Z} M_z$ the direct sum of M_z's which is an ideal of $\prod_{z\in Z} M_z$.

Definition 8. *For each $z \in Z$, we fix a matrix unit $\{e^z_{i,j}\}^2_{i,j=1}$ of $M_z = \mathbb{M}_2$. For each $x \in X$, we define a projection $p_x \in \prod_{z\in Z} M_z$ by*

$$p_x(z) = \begin{cases} e^z_{1,1} & \text{if } x \in z, \\ 0 & \text{if } x \notin z. \end{cases}$$

We denote by A the C^*-subalgebra of $\prod_{z\in Z} M_z$ generated by $\bigoplus_{z\in Z} M_z$ and $\{p_x\}_{x\in X}$.

Definition 9. *For each $z = \{x_1, x_2\} \in Z$, we fix a matrix unit $\{e^z_{x_i,x_j}\}^2_{i,j=1}$ of $M_z = \mathbb{M}_2$. For each $x \in X$, we define a projection $q_x \in \prod_{z\in Z} M_z$ by*

$$q_x(z) = \begin{cases} e^z_{x,x} & \text{if } x \in z, \\ 0 & \text{if } x \notin z. \end{cases}$$

We denote by B the C^*-subalgebra of $\prod_{z\in Z} M_z$ generated by $\bigoplus_{z\in Z} M_z$ and $\{q_x\}_{x\in X}$.

The following easy lemma illustrates an difference of A and B.

Lemma 10. *For $x, y \in X$ with $x \neq y$, we have $p_x p_y = e^{\{x,y\}}_{1,1} \neq 0$, and $q_x q_y = 0$.*

Proof. Straightforward.

Definition 11. *Let λ be a finite subset of X. We denote by A_λ the C^*-subalgebra of A spanned by $\bigoplus_{z\subset\lambda} M_z$ and $\{p_x\}_{x\in\lambda}$, and by B_λ the C^*-subalgebra of B spanned by $\bigoplus_{z\subset\lambda} M_z$ and $\{q_x\}_{x\in\lambda}$,*

Lemma 12. *There exist isomorphisms*

$$A_\lambda \cong B_\lambda \cong \bigoplus_{z\subset\lambda} \mathbb{M}_2 \oplus \bigoplus_{x\in\lambda} \mathbb{C}$$

for each finite set $\lambda \subset X$ such that two inclusions $A_\lambda \subset A_\mu$ and $B_\lambda \subset B_\mu$ have the same multiplicity.

Proof. For $x \in \lambda$, let us denote $p'_x \in A_\lambda$ by $p'_x = p_x - \sum_{y \in \lambda \setminus \{x\}} e_{1,1}^{\{x,y\}}$. Then we have an orthogonal decomposition

$$A_\lambda = \sum_{z \subset \lambda} M_z + \sum_{x \in \lambda} \mathbb{C} p'_x.$$

This proves $A_\lambda \cong \bigoplus_{z \subset \lambda} \mathbb{M}_2 \oplus \bigoplus_{x \in \lambda} \mathbb{C}$. Similarly we have $B_\lambda \cong \bigoplus_{z \subset \lambda} \mathbb{M}_2 \oplus \bigoplus_{x \in \lambda} \mathbb{C}$. Now it is routine to check the last statement.

Proposition 13. *Two C^*-algebras A and B are AF-algebras, and the directed families $\{A_\lambda\}$ and $\{B_\lambda\}$ of finite dimensional C^*-subalgebras give isomorphic Bratteli diagrams.*

Proof. Follows from the facts

$$\overline{\bigcup_{\lambda \subset X} A_\lambda} = A, \quad \overline{\bigcup_{\lambda \subset X} B_\lambda} = B$$

and Lemma 12.

Remark 14. From Proposition 13, we can show that $K_0(A)$ and $K_0(B)$ are isomorphic as scaled ordered groups. In fact, they are isomorphic to the subgroup G of $\prod_{z \in Z} \mathbb{Z}$ generated by $\bigoplus_{z \in Z} \mathbb{Z}$ and $\{g_x\}_{x \in X}$, where $g_x \in \prod_{z \in Z} \mathbb{Z}$ is defined by

$$g_x(z) = \begin{cases} 1 & \text{if } x \in z, \\ 0 & \text{if } x \notin z. \end{cases}$$

The order of G is the natural one, and its scale is

$$\{g \in G \mid 0 \leq g(z) \leq 2 \text{ for all } z \in Z\}.$$

From this fact and Elliott's theorem (Theorem 6.4 of [3]), we can show the next lemma, although we give a direct proof here.

Proposition 15. *When X is countable, A and B are isomorphic.*

Proof. Let us list $X = \{x_1, x_2, \ldots\}$. We define a $*$-homomorphism $\varphi \colon A \to B$ as follows. For $z = \{x_k, x_l\}$, we define $\varphi(e_{i,j}^z) = e_{x_{n_i}, x_{n_j}}^z$ where $n_1 = k, n_2 = l$ when $k < l$ and $n_1 = l, n_2 = k$ when $k > l$. For $x_k \in X$, we set

$$\varphi(p_{x_k}) = q_{x_k} + \sum_{i=1}^{k-1} \left(e_{x_i, x_i}^{\{x_i, x_k\}} - e_{x_k, x_k}^{\{x_i, x_k\}} \right).$$

Now it is routine to check that φ is an isomorphism from A to B.

Proposition 15 is no longer true for uncountable X. To see this, we need the following lemma.

Lemma 16. *There exists a surjection* $\pi_A \colon A \to \bigoplus_{x\in X} \mathbb{C}$ *defined by* $\pi_A(M_z) = 0$ *for* $z \in Z$ *and* $\pi_A(p_x) = \delta_x$ *for* $x \in X$. *Its kernal is* $\bigoplus_{z\in Z} M_z$ *which coincides with the ideal generated by the all commutators* $xy - yx$ *of* A. *The same is true for* B.

Proof. Let π_A be the quotient map from A to $A/\bigoplus_{z\in Z} M_z$. Then $A/\bigoplus_{z\in Z} M_z$ is generated by $\{\pi_A(p_x)\}_{x\in X}$ which is an orthogonal family of non-zero projections. This proves the first statement. Since $\bigoplus_{x\in X} \mathbb{C}$ is commutative, the ideal $\bigoplus_{z\in Z} M_z$ contains all commutators. Conversely, the ideal generated by the commutators of A contains $\bigoplus_{z\in Z} M_z$ because $\mathbb{M}_2$ is generated by its commutators. This shows that $\bigoplus_{z\in Z} M_z$ is the ideal generated by the all commutators of A. The proof goes similarly for B.

Proposition 17. *When* X *is uncountable,* A *and* B *are not isomorphic.*

Proof. To the contrary, suppose that there exists an isomorphism $\varphi \colon A \to B$. By Lemma 16, $\bigoplus_{z\in Z} M_z$ is the ideal generated by the all commutators in both A and B. Hence φ preserves this ideal $\bigoplus_{z\in Z} M_z$. Thus we get the following commutative diagram with exact rows;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{z\in Z} M_z & \longrightarrow & A & \xrightarrow{\pi_A} & \bigoplus_{x\in X} \mathbb{C} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus_{z\in Z} M_z & \longrightarrow & B & \xrightarrow{\pi_B} & \bigoplus_{x\in X} \mathbb{C} & \longrightarrow & 0. \end{array}$$

Since the family of projections $\{q_x\}_{x\in X}$ in B is mutually orthogonal, the surjection $\pi_B \colon B \to \bigoplus_{x\in X} \mathbb{C}$ has a splitting map $\sigma_B \colon \bigoplus_{x\in X} \mathbb{C} \to B$ defined by $\sigma_B(\delta_x) = q_x$. Hence by the diagram above, the surjection $\pi_A \colon A \to \bigoplus_{x\in X} \mathbb{C}$ also has a splitting map $\sigma_A \colon \bigoplus_{x\in X} \mathbb{C} \to A$. Let us set $p'_x = \sigma_A(\delta_x)$ for $x \in X$. Choose a countable infinite subset Y of X. For each $y \in Y$, the set

$$\mathcal{F}_y = \bigl\{x \in X \bigm| x \neq y, \|(p_y - p'_y)(\{x, y\})\| \geq 1/2\bigr\}$$

is finite, because $p_y - p'_y \in \ker \pi_A = \bigoplus_{z\in Z} M_z$. Since X is uncountable, we can find $x_0 \in X$ with $x_0 \notin Y \cup \bigcup_{y\in Y} \mathcal{F}_y$. Since

$$\mathcal{F}_{x_0} = \bigl\{x \in X \bigm| x \neq x_0, \|(p_{x_0} - p'_{x_0})(\{x, x_0\})\| \geq 1/2\bigr\}$$

is finite, we can find $y_0 \in Y \setminus \mathcal{F}_{x_0}$. We set $z = \{x_0, y_0\}$. From $y_0 \notin \mathcal{F}_{x_0}$, we have $\|(p_{x_0} - p'_{x_0})(z)\| < 1/2$, and from $x_0 \notin \mathcal{F}_{y_0}$, we have $\|(p_{y_0} - p'_{y_0})(z)\| < 1/2$. However, $p_{x_0}(z) = p_{y_0}(z) = e^z_{1,1}$ and $p'_{x_0}(z)$ is orthogonal to $p'_{y_0}(z)$. This is a contradiction. Thus A and B are not isomorphic.

Combining Proposition 13 and Proposition 17, we get Theorem 6.

3 A Prime AF-Algebra Which is Not Primitive

In this section, we construct an AF-algebra which is prime but not primitive. Although we follow the idea of Weaver in [6], our construction of the C^*-algebra and proof of the main theorem is much easier than the ones there. A similar construction can be found in [4], but the proof there uses general facts of topological graph algebras.

Let X be an uncountable set, and Λ be the directed set of all finite subsets of X. For $n \in \mathbb{N}$, we set $\Lambda_n = \{\lambda \subset X \mid |\lambda| = n\}$. We get $\Lambda = \coprod_{n=0}^{\infty} \Lambda_n$.

Definition 18. *For $n \in \mathbb{Z}_+$ and $\lambda \in \Lambda_n$, we define*

$$l(\lambda) = \{t \colon \{1, \dots, n\} \to \lambda \mid t \text{ is a bijection}\}.$$

For $\emptyset \in \Lambda$, we define $l(\emptyset) = \{\emptyset\}$.

Note that $|l(\lambda)| = n!$ for $\lambda \in \Lambda_n$ and $n \in \mathbb{N}$.

Definition 19. *For $n \in \mathbb{N}$ and $\lambda \in \Lambda_n$, we define $M_\lambda \cong \mathbb{M}_{n!}$ whose matrix unit is given by $\{e_{s,t}^{(\lambda)}\}_{s,t \in l(\lambda)}$.*

Definition 20. *Take $\lambda \in \Lambda_n$ and $\mu \in \Lambda_m$ with $\lambda \cap \mu = \emptyset$. For $t \in l(\lambda)$ and $s \in l(\mu)$, we define $ts \in l(\lambda \cup \mu)$ by*

$$(ts)(i) = \begin{cases} t(i) & \text{for } i = 1, \dots, n \\ s(i-n) & \text{for } i = n+1, \dots, n+m. \end{cases}$$

Note that when $\mu = \emptyset$, we have $t\emptyset = t$.

Definition 21. *For $\lambda, \mu \in \Lambda$ with $\lambda \subset \mu$, we define a $*$-homomorphism $\iota_{\mu,\lambda} \colon M_\lambda \to M_\mu$ by*

$$\iota_{\mu,\lambda}\big(e_{s,t}^{(\lambda)}\big) = \sum_{u \in l(\mu \setminus \lambda)} e_{su,tu}^{(\mu)} \quad \text{for } s, t \in l(\lambda).$$

Note that $\iota_{\lambda,\lambda}$ is the identity map of M_λ, and that $\iota_{\lambda_3,\lambda_2} \circ \iota_{\lambda_2,\lambda_1} \neq \iota_{\lambda_3,\lambda_1}$ for $\lambda_1 \subsetneq \lambda_2 \subsetneq \lambda_3$. For $\lambda_1, \lambda_2 \in \Lambda_n$ and $\mu \in \Lambda_m$ with $\lambda_1 \neq \lambda_2$ and $\lambda_1 \cup \lambda_2 \subset \mu$, the images $\iota_{\mu,\lambda_1}(M_{\lambda_1})$ and $\iota_{\mu,\lambda_2}(M_{\lambda_2})$ are mutually orthogonal.

Definition 22. *For $\lambda \in \Lambda$, we define a $*$-homomorphism $\iota_\lambda \colon M_\lambda \to \prod_{\mu \in \Lambda} M_\mu$ by*

$$\iota_\lambda(x)(\mu) = \begin{cases} \iota_{\mu,\lambda}(x) & \text{if } \lambda \subset \mu, \\ 0 & \text{otherwise,} \end{cases}$$

for $x \in M_\lambda$. We set $N_\lambda = \iota_\lambda(M_\lambda) \subset \prod_{\mu \in \Lambda} M_\mu$ and $f_{s,t}^{(\lambda)} = \iota_\lambda(e_{s,t}^{(\lambda)}) \in N_\lambda$ for $s, t \in l(\lambda)$.

For $\lambda \in \Lambda_n$, We have $N_\lambda \cong \mathbb{M}_{n!}$ and $\{f^{(\lambda)}_{s,t}\}_{s,t\in l(\lambda)}$ is a matrix unit of N_λ.

Lemma 23. *For $\lambda, \mu \in \Lambda$ with $\lambda \subset \mu$, $s, t \in l(\lambda)$ and $s', t' \in l(\mu)$, we have $f^{(\lambda)}_{s,t} f^{(\mu)}_{s',t'} = f^{(\mu)}_{su,t'}$ when $s' = tu$ with some $u \in l(\mu \setminus \lambda)$, and $f^{(\lambda)}_{s,t} f^{(\mu)}_{s',t'} = 0$ otherwise.*

Proof. Straightforward.

Lemma 24. *For $\lambda, \mu \in \Lambda$, we have $0 \neq N_\lambda N_\mu \subset N_\mu$ if $\lambda \subset \mu$, $0 \neq N_\lambda N_\mu \subset N_\lambda$ if $\lambda \supset \mu$, and $N_\lambda N_\mu = 0$ otherwise.*

Proof. If $\lambda \subset \mu$, we have $0 \neq N_\lambda N_\mu \subset N_\mu$ by Lemma 23. Similarly we have $0 \neq N_\lambda N_\mu \subset N_\lambda$ if $\lambda \supset \mu$. Otherwise, we can easily see $N_\lambda N_\mu = 0$ from the definition.

Corollary 25. *For each n, the family $\{N_\lambda\}_{\lambda\in\Lambda_n}$ of C^*-algebras is mutually orthogonal.*

Corollary 26. *Take $\lambda, \lambda' \in \Lambda$ with $\lambda \subset \lambda'$. Let $p_{\lambda'}$ be the unit of $N_{\lambda'}$. Then $N_\lambda \ni a \mapsto ap_{\lambda'} \in N_{\lambda'}$ is an injective $*$-homomorphism.*

Definition 27. *We define $A = \overline{\sum_{\lambda\in\Lambda} N_\lambda} \subset \prod_{\mu\in\Lambda} M_\mu$.*

Proposition 28. *The set A is an AF-algebra.*

Proof. For each $\mu \in \Lambda$, $A_\mu = \sum_{\lambda\subset\mu} N_\lambda$ is a finite dimensional C^*-algebra by Lemma 24. For $\lambda, \mu \in \Lambda$ with $\lambda \subset \mu$, we have $A_\lambda \subset A_\mu$. Hence $A = \overline{\bigcup_{\mu\in\Lambda} A_\mu}$ is an AF-algebra.

Lemma 29. *Every non-zero ideal I of A contains N_λ for some $\lambda \in \Lambda$.*

Proof. As in the proof of Proposition 28, we set $A_\mu = \sum_{\lambda\subset\mu} N_\lambda$ for $\mu \in \Lambda$. Since $A = \overline{\bigcup_{\mu\in\Lambda} A_\mu}$, we have $I = \overline{\bigcup_{\mu\in\Lambda}(I \cap A_\mu)}$ for an ideal I of A. Hence if I is nonzero, we have $I \cap A_{\mu_0} \neq 0$ for some $\mu_0 \in \Lambda$. Thus we can find a non-zero element $a \in I$ in the form $a = \sum_{\lambda\subset\mu_0} a_\lambda$ for $a_\lambda \in N_\lambda$. Since $a \neq 0$, we can find $\lambda_0 \in \Lambda$ with $\lambda_0 \subset \mu_0$ such that $a_{\lambda_0} \neq 0$ and $a_\lambda = 0$ for all $\lambda \subsetneq \lambda_0$. Take $x_0 \in X$ with $x_0 \notin \mu_0$. Set $\lambda'_0 = \lambda_0 \cup \{x_0\}$. Let $p_{\lambda'_0}$ be the unit of $N_{\lambda'_0}$. For $\lambda \subset \mu_0$, $a_\lambda p_{\lambda'_0} \neq 0$ only when $\lambda \subset \lambda_0$. Hence we have $ap_{\lambda'_0} = a_{\lambda_0} p_{\lambda'_0}$. By Corollary 26, $a_{\lambda_0} p_{\lambda'_0}$ is a non-zero element of $N_{\lambda'_0}$. Hence we can find a non-zero element in $I \cap N_{\lambda'_0}$. Since $N_{\lambda'_0}$ is simple, we have $N_{\lambda'_0} \subset I$. We are done.

Lemma 30. *If an ideal I of A satisfies $N_{\lambda_0} \subset I$ for some $\lambda_0 \in \Lambda$, then $N_\lambda \subset I$ for all $\lambda \supset \lambda_0$.*

Proof. Clear from Lemma 24 and the simplicity of N_λ.

Proposition 31. *The C^*-algebra is prime but not primitive.*

Proof. Take two non-zero ideals I_1, I_2 of A. By Lemma 29, we can find $\lambda_1, \lambda_2 \in \Lambda$ such that $N_{\lambda_1} \subset I_1$ and $N_{\lambda_2} \subset I_2$. Set $\lambda = \lambda_1 \cup \lambda_2 \in \Lambda$. By Lemma 30, we have $N_\lambda \subset I_1 \cap I_2$. Thus $I_1 \cap I_2 \neq 0$. This shows that A is prime.

To prove that A is not primitive, it suffices to see that for any state φ of A we can find a non-zero ideal I such that $\varphi(I) = 0$ (see [6]). Take a state φ of A. By Corollary 25, the family $\{N_\lambda\}_{\lambda \in \Lambda_n}$ of C^*-algebras is mutually orthogonal for each $n \in \mathbb{N}$. Hence the set

$$\Omega_n = \{\lambda \in \Lambda_n \mid \text{the restriction of } \varphi \text{ to } N_\lambda \text{ is non-zero}\}$$

is countable for each $n \in \mathbb{N}$. Since X is uncountable, we can find $x_0 \in X$ such that $x_0 \notin \lambda$ for all $\lambda \in \bigcup_{n \in \mathbb{N}} \Omega_n$. Let $I = \overline{\sum_{\lambda \ni x_0} N_\lambda}$. Then I is an ideal of A by Lemma 24. Since $\lambda \ni x_0$ implies $\varphi(N_\lambda) = 0$, we have $\varphi(I) = 0$. Therefore A is not primitive.

This finishes the proof of Theorem 7.

Remark 32. Let $(A_\lambda, \varphi_{\mu,\lambda})$ be an inductive system of finite dimensional C^*-algebras over a directed set Λ, and A be its inductive limit. It is not hard to see that the AF-algebra A is prime if and only if the Bratteli diagram of the inductive system satisfies the analogous condition of (iii) in Corollary 3.9 of [1]. Hence, the Bratteli diagram of an inductive system of finite dimensional C^*-algebras determines the primeness of the inductive limit, although it does not determine the inductive limit itself. However the primitivity of the inductive limit is not determined by the Bratteli diagram. In fact, in a similar way to the construction of Example 5, we can find an inductive system of finite dimensional C^*-algebras whose Bratteli diagram is isomorphic to the one coming from the directed family $\{A_\lambda\}$ constructed in the proof of Proposition 28, but the inductive limit is separable. This AF-algebra is primitive because it is separable and prime (see, for example, Proposition 4.3.6 of [5]).

References

1. Bratteli, O. *Inductive limits of finite dimensional C^*-algebras.* Trans. Amer. Math. Soc. 171 (1972), 195–234.
2. Dixmier, J. *On some C^*-algebras considered by Glimm.* J. Funct. Anal. **1** (1967) 182–203.
3. Elliott, G. A. *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras.* J. Algebra **38** (1976), no. 1, 29–44.
4. Katsura, T. *A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras III, ideal structures.* Ergodic Theory Dynam. Systems (to appear).
5. Pedersen, G. K. *C^*-algebras and their automorphism groups.* London Mathematical Society Monographs, **14**. Academic Press, Inc., London-New York, 1979.
6. Weaver, N. *A prime C^*-algebra that is not primitive.* J. Funct. Anal. **203** (2003), no. 2, 356–361.

Central Sequences in C^*-Algebras and Strongly Purely Infinite Algebras

Eberhard Kirchberg

Institut für Mathematik, Humboldt Universität zu Berlin,
Unter den Linden 6, D–10099 Berlin, Germany,
kirchbrg@mathematik.hu-berlin.de

Summary. It is shown that $F(A) := (A' \cap A_\omega)/\mathrm{Ann}(A, A_\omega)$ is a unital C^*-algebra and that $A \mapsto F(A)$ is a stable invariant of separable C^*-algebras A with certain local continuity and permanence properties. Here A_ω means the ultrapower of A.

If A is separable, then $F(A)$ is simple, if and only if, either $A \otimes \mathcal{K} \cong \mathcal{K}$ or A is a simple purely infinite *nuclear* C^*-algebra. In the first case $F(A) \cong \mathbb{C}$, and in the second case $F(A)$ is purely infinite and A absorbs $\mathcal{O}_\infty$ tensorially, i.e. $A \cong A \otimes \mathcal{O}_\infty$.

We show that $F(Q) = \mathbb{C} \cdot 1$ for the Calkin algebra $Q := \mathcal{L}/\mathcal{K}$, in contrast to the separable case.

We introduce a "locally semi-projective" invariant $\mathrm{cov}(B) \in \mathbb{N} \cup \{\infty\}$ of unital C^*-algebras B with $\mathrm{cov}(B) \leq \mathrm{cov}(C)$ if there is a unital *-homomorphism from C into B. If B is nuclear and has no finite-dimensional quotient then $\mathrm{cov}(B) \leq \mathrm{dr}(B)+1$ for the decomposition rank $\mathrm{dr}(B)$ of B. (Thus, $\mathrm{cov}(\mathcal{Z}) = 2$ for the Jian–Su algebra $\mathcal{Z}$.) Separable (not necessarily simple) C^*-algebras A are strongly purely infinite in the sense of [25] if A does not admit a non-trivial lower semi-continuous 2-quasi-trace and $F(A)$ contains a simple C^*-subalgebra B with $\mathrm{cov}(B) < \infty$ and $1 \in B$. In particular, $A \otimes \mathcal{Z}$ is strongly purely infinite if A_+ admits no non-trivial lower semi-continuous 2-quasi-trace.

Properties of $F(A)$ will be used to show that A is tensorially $\mathcal{D}$–absorbing, (i.e. that $A \otimes \mathcal{D} \cong A$ by an isomorphism that is approximately unitarily equivalent to $a \mapsto a \otimes 1$), if A is stable and separable, $\mathcal{D}$ is a unital tensorially self-absorbing algebra, and $\mathcal{D}$ is unitally contained in $F(A)$. It follows that the class of tensorially $\mathcal{D}$–absorbing separable stable C^*-algebras A, is closed under inductive limits and passage to ideals and quotients. The local permanence properties of the functor $A \mapsto F(A)$ imply that this class is also closed under extensions, if and only if, every commutator uvu^*v^* of unitaries $u, v \in \mathcal{U}(\mathcal{D})$ is contained in the connected component $\mathcal{U}_0(\mathcal{D})$ of 1 in $\mathcal{U}(\mathcal{D})$. If this is the case, then the class of (not necessarily stable) $\mathcal{D}$–absorbing separable C^*-algebras is also closed under passage to hereditary C^*-algebras.

1 Introduction: The Stable Invariant $F(A)$.

The different results of this paper (stated in the summary) will be derived from properties of the relative commutant A^c of a C^*-algebra A in its ultrapower. Our considerations suggest that a study of the ideals and simple C^*-subalgebras of the below defined quotient algebra $F(A)$ of A^c could be useful. There are related open problems: the UCT problem, the classification of (strongly tensorially) self-absorbing C^*-algebras $\mathcal{D}$, permanence properties for *all* $\mathcal{D}$-absorbing algebras, the question which additional properties imply that purely infinite algebras are strongly purely infinite, and to the existence of certain asymptotic algebras suitable for a KK-theoretic formulation of the classification of $\mathcal{D}$-absorbing algebras.

Note that our technics is *not* a sort of non-standard analysis: All appearing algebras are honest C^*-algebras over $\mathbb{C}$ and all considered maps between them are at least completely positive maps. We consider here only $A \mapsto F(A)$ for a fixed free ultra-filter ω on $\mathbb{N}$, because we hope that it is helpful for the reader to get an impression of what we consider as asymptotic analysis of C^*-algebras if $\mathbb{N}$ is replaced e.g. by $\mathbb{R}_+$. There are surprising relations between algebraic properties of $F(A)$ and analytic properties of separable A. See e.g. Lemmas 2.8, 2.11(3), Propositions 1.17, 4.11, Corollary 1.13 (in view of applications), and Theorems 2.12, 3.10, 4.5.

Let ω a *free* ultra-filter on $\mathbb{N}$. We also denote by ω the related *character* on $\ell_\infty := \ell_\infty(\mathbb{N})$ with $\omega(c_0(\mathbb{N})) = \{0\}$. Recall that $\lim_\omega \alpha_n$ means the complex number $\omega(\alpha_1, \alpha_2, \ldots)$ for $(\alpha_1, \alpha_2, \ldots) \in \ell_\infty$. For a C^*-algebra A, we let

$$c_\omega(A) := \{(a_1, a_2, \ldots) \in \ell_\infty(A)\,;\; \lim_\omega \|a_n\| = 0\}\,,$$
$$A_\omega := \ell_\infty(A)/c_\omega(A)$$

A_ω will be called the *ultrapower* of A. The natural epimorphism from $\ell_\infty(A)$ onto A_ω will be denoted by π_ω. $(a_1, a_2, \ldots) \in \ell_\infty(A)$ is a *representing sequence* for $b \in A_\omega$ if $\pi_\omega(a_1, a_2, \ldots) = b$. We consider A as a C^*-subalgebra of A_ω by the diagonal embedding

$$a \mapsto \pi_\omega(a, a, \ldots) = (a, a, \ldots) + c_\omega(A).$$

Then $A^c := A' \cap A_\omega$ is the *algebra of (ω-) central sequences in* A (modulo ω-zero sequences). It is easy to see that the (two-sided) *annihilator*

$$\mathrm{Ann}(A) := \mathrm{Ann}(A, A_\omega) := \{b \in A_\omega\,;\; bA = \{0\} = Ab\}$$

of A in A_ω is an ideal of A^c. We let

$$F(A) := A^c/\mathrm{Ann}(A) = (A' \cap A_\omega)/\mathrm{Ann}(A, A_\omega)$$

It turns out that $F(A)$ is unital for σ-unital A, and that $A \mapsto F(A)$ is an invariant of Morita equivalence classes of σ-unital C^*-algebras. We generalize A^c and $F(A)$ for C^*-subalgebras $A \subset \mathcal{M}(B)_\omega$ to get more flexible tools for the proofs of permanence properties:

Definitions 1.1 *Suppose that B is a C^*-algebra, $\mathcal{M}(B)$ its multiplier algebra, and that A is a C^*-subalgebra of $\mathcal{M}(B)_\omega$. We let, for $A \subset \mathcal{M}(B)_\omega$,*

$$\begin{aligned} (A,B)^c &:= A' \cap B_\omega \subset A' \cap \mathcal{M}(B)_\omega\,, \\ \operatorname{Ann}(A,B_\omega) &:= \{b \in B_\omega\,;\ Ab + bA = \{0\}\} \\ F(A,B) &:= (A,B)^c / \operatorname{Ann}(A,B_\omega)\,, \\ D_{A,B} &:= \overline{\operatorname{span}(AB_\omega A)} \subset B_\omega \quad \text{and} \\ \mathcal{N}(D_{A,B}) = \mathcal{N}(D_{A,B},B_\omega) &:= \{b \in B_\omega\,;\ bD_{A,B} + D_{A,B}b \subset D_{A,B}\}\,. \end{aligned}$$

*We denote by $\rho_{A,B}$ the natural *-morphism*

$$\rho_{A,B}\colon\ F(A,B) \otimes^{\max} A \to D_{A,B} \subset B_\omega$$

given by $\rho_{A,B}((b + \operatorname{Ann}(A,B_\omega)) \otimes a) := ba$ for $b \in (A,B)^c$ and $a \in A$.

The Definitions of $F(A,B)$ and of $\rho_{A,B}$ make sense, because (obviously) $\operatorname{Ann}(A,B_\omega)$ is a closed ideal of $(A,B)^c$, $(A,B)^c$ and A commute element-wise and $A \cdot \operatorname{Ann}(A,B_\omega) = \{0\}$.

Then $F(A) = F(A,A)$, $(A,B)^c = (A,\mathcal{M}(B))^c \cap B_\omega$ is a closed ideal of $(A,\mathcal{M}(B))^c$ and $\operatorname{Ann}(A,B_\omega) = \operatorname{Ann}(D_{A,B},B_\omega) = \operatorname{Ann}(A,\mathcal{M}(B)_\omega) \cap B_\omega$. We write $\mathcal{N}(D_{A,B})$ for $\mathcal{N}(D_{A,B},B_\omega)$, D_A, for $D_{A,A}$, $\mathcal{N}(D_A)$ for $\mathcal{N}(D_{A,A})$, ρ_A or ρ for $\rho_{A,A}$, ... and so on.

Let $\mathcal{K}$ denote the compact operators on $\ell_2(\mathbb{N})$. $\mathcal{K}^c = \operatorname{Ann}(\mathcal{K}) + \mathbb{C}\cdot 1$ is huge, but $F(\mathcal{K}) \cong \mathbb{C} = \mathbb{C}_\omega$ (cf. Corollary 1.10). Permanence properties of $F(A)$ have to be considered with some care, because e.g. $F(\mathcal{K} + \mathbb{C}\cdot 1) \cong (\mathcal{K} + \mathbb{C}\cdot 1)^c = \operatorname{Ann}(\mathcal{K}) + \mathbb{C}\cdot 1$.

The below given basic facts on $(A,B)^c$, $\operatorname{Ann}(A,B_\omega)$ and $F(A,B)$ will be proved in Appendix B or are taken from [22, sec. 2.2].

Definition 1.2 *A convex subcone $\mathcal{V} \subset CP(B,C)$ of the cone of completely positive (=:c.p.) maps from B in C is (matricially)* operator-convex *if the c.p. map $b \mapsto c^*V(r^*br)c$ is in $\mathcal{V}$ for every $V \in \mathcal{V}$ and every row $r \in M_{1,n}(B)$ and column $c \in M_{n,1}(C)$.*

Examples of operator-convex cones are the cone $CP_{nuc}(B,C)$ of nuclear c.p. maps from B into C and the cone $CP_{fin}(B,C)$ of the c.p. maps of finite rank. If $B \subset \mathcal{M}(C)$ then the cone of approximately inner c.p. maps $V \to B \to C$ is operator-convex.

Proposition 1.3 *Suppose that $A \subset B_\omega$ is separable, that $\mathcal{V} \subset CP(B,B)$ is an operator-convex cone of completely positive maps from B into B, that $J \subset B$ is a closed ideal, and that $a \in A' \cap B_\omega$, $b, c \in B_\omega$ are positive contractions with $ab = ac = bc = 0$ and $bAc = \{0\}$.*
If $c \in J_\omega \subset B_\omega$, and if there is a bounded sequence $S_1, S_2, \ldots \in \mathcal{V}$ such that $S_\omega(x) = b^*xb$ for $x \in A$, then there are positive contractions $e, f, g \in A' \cap B_\omega$ and a sequence of contractions $T_1, T_2, \ldots \in \mathcal{V}$ with

(1) $ea = a$, $fb = b$, $gc = c$ *and* $ef = eg = fg = 0$
(2) $T_\omega(x) = xf$ *for all* $x \in A$.
(3) $g \in J_\omega$

We use only particular aspects of this Proposition, e.g., where at least one of the a, b, c is zero. Note that the assumption on $(\mathcal{V}, b)$ is trivially satisfied for $\mathcal{V} = CP(B, B)$ or for the operator-convex cone $\mathcal{V}$ of all inner c.p. maps (and the conclusion (2) is then trivial, too). The same happens with the assumptions on (J, c) if we let $J = B$.

Part (2) shows that (ultapowers of) *operator convex cones* $\mathcal{V} \subset CP(B, B)$ *define in a natural way closed ideals of* $F(A, B)$ (compare the proof of Lemma 2.11).

Definition 1.4 *We call a C^*-algebra C* σ-sub-Stonean *if for every separable C^*-subalgebra $A \subset C$ and every $b, c \in C_+$ with $bc = 0$ and $bAc = \{0\}$ there are positive contractions $f, g \in A' \cap C$ with $fg = 0$, $fb = b$ and $gc = c$.*

Obviously, if C is σ-sub-Stonean, then C is sub-Stonean (which is the case $A = \{0\}$), and $B' \cap C$ is σ-sub-Stonean for every separable C^*-subalgebra B of C (consider $C^*(B, A)$ in place of A in the definition). It is easy to see, that if D is a hereditary C^*-subalgebra of C, then D is σ-sub-Stonean if and only if for every $a \in D_+$ there is a positive contraction $e \in D$ with $ea = e$. In particular, $\mathrm{Ann}(d, C)$ is σ-sub-Stonean for every $d \in C_+$ if C is σ-sub-Stonean. Further, if C is σ-sub-Stonean and $I \triangleleft C$ is a σ-sub-Stonean closed ideal of C, then C/I is σ-sub-Stonean. (An exercise.)

Definitions 1.5 *We call a closed ideal I of a C^*-algebra C a* σ-ideal of C *if for every separable C^*-subalgebra $A \subset C$ and every $d \in I_+$ there is a positive contraction $e \in A' \cap I$ with $ed = d$.*

We say that a short exact sequence of C^*-algebras $0 \to B \to C \to D \to 0$ (with epimorphism $\pi\colon C \to D$) is strongly locally semi-split *if for every separable C^*-subalgebra $A \subset D$ there is a *-morphism ψ from $C_0((0,1], A) \cong C_0(0,1] \otimes A$ into C such that $\pi \circ \psi(f_0 \otimes a) = a$, where $f_0(t) = t$ for $t \in (0, 1]$.*

Note that $A' \cap I$ is σ-sub-Stonean if C is σ-sub-Stonean, $I \triangleleft C$ is a σ-ideal and A is a separable.

One can see, that $A' \cap I$ is a σ-ideal of $A' \cap C$ and is a non-degenerate C^*-subalgebra of I.

It is easy to see, that the image $\varphi(I)$ is σ-ideal of $\varphi(C)$ for every morphism $\varphi\colon C \to E$. Furthermore, if $I \subset C \subset E$ and I is a closed σ-ideal of E, then I is also a σ-ideal of C. Clearly, the intersection and sum of two σ-ideals is a σ-ideal.

An elementary consequence of the definitions is given by:

Proposition 1.6 *If I is a σ-ideal of a C^*-algebra C, then, for every separable C^*-subalgebra $A \subset C$, $A' \cap I$ is a non-degenerate C^*-subalgebra of I, $\pi_I(\mathrm{Ann}(A, I)) = \mathrm{Ann}(\pi_I(A), C/I)$ and the sequence*

$$0 \to A' \cap I \to A' \cap C \to \pi_I(A)' \cap (C/I) \to 0$$

is exact and strongly locally semi-split.

The epimorphism $A' \cap C \to \pi_I(A)' \cap (C/I)$ is the restriction of the natural epimorphism π_I from C onto C/I.

Proposition 1.3 and the above discussed permanence properties for σ-sub-Stonean algebras and σ-ideals imply:

Corollary 1.7 *Suppose that J is a closed ideal of B and that A is a separable C^*-subalgebra of B_ω.*

Then B_ω, $(A,B)^c$, $\mathrm{Ann}(A,B_\omega)$, and $F(A,B)$ are σ-sub-Stonean.

J_ω, $J_\omega \cap (A,B)^c$ and $\mathrm{Ann}(A,B_\omega)$ are σ-ideals of B_ω respectively of (A,B).

In particular, B_ω, $\mathrm{Ann}(A,B_\omega)$, $\mathrm{Ann}(A)$, $(A,B)^c$, A^c, $F(A,B)$, $F(A)$, $J_\omega \cap (A,B)^c$ and $J_\omega \cap \mathrm{Ann}(A,B_\omega)$ are sub–Stonean.

The permanence properties for σ-ideals imply e.g. that $J_\omega \cap \mathrm{Ann}(A,B_\omega)$ is a σ-ideal in $(A,B)^c$ and $\mathrm{Ann}(A,B_\omega)$.

By Proposition 1.6 the statement that $\mathrm{Ann}(A,B_\omega)$ is a σ-ideal of $(A,B)^c$ implies:

Corollary 1.8 *Suppose that A is a separable C^*-subalgebra of B_ω, and that C is a separable C^*-subalgebra of $F(A,B)$. There is a *-morphism*

$$\lambda\colon\ C_0((0,1],C) \to (A,B)^c = A' \cap B_\omega$$

with $\lambda(f) + \mathrm{Ann}(A,B_\omega) = f(1) \in C \subset F(A,B)$ for $f \in C_0((0,1],C)$.

The following proposition gives some elementary properties of $(A,B)^c$, $\mathrm{Ann}(A,B)$ and $F(A,B)$.

Proposition 1.9 *Suppose that A is a σ-unital C^*-subalgebra of B_ω, and let $D_{A,B}$, $\mathcal{N}(D_{A,B})$, $\mathrm{Ann}(A,B_\omega)$, $(A,B)^c$, $F(A,B)$ and $\rho_{A,B}$ be as in Definitions 1.1. Then*

(1) $\mathrm{Ann}(A,B_\omega)$ *is an ideal of $\mathcal{N}(D_{A,B})$, and*

$$\mathrm{Ann}(A,B_\omega) \subset (A,B)^c \subset \mathcal{N}(D_{A,B})\,.$$

(2) For every countable subset $Y \subset B_\omega$ there exists a positive contraction $e \in (A,B)^c$ with $ey = ye = y$ for all $y \in Y$.

(3) $F(A,B)$ is unital. Moreover, if $a_0 \in A_+$ a strictly positive element of A and $e \in B_\omega$ is a positive contraction, then e satisfies $ea_0 = a_0 = a_0e$, if and only if, $e \in (A,B)^c$ and $e + \mathrm{Ann}(A,B_\omega) = 1$ *in $F(A,B)$.*

*(4) The natural *-morphism $\mathcal{N}(D_{A,B}) \to \mathcal{M}(D_{A,B})$ is an* epimorphism *onto $\mathcal{M}(D_{A,B})$ with kernel* $= \mathrm{Ann}(A,B_\omega)$.

*(5) The epimorphism from $\mathcal{N}(D_{A,B})$ onto $\mathcal{M}(D_{A,B})$ defines a *-isomorphism η from $F(A,B)$ onto $A' \cap \mathcal{M}(D_{A,B})$ with $\rho_{A,B}(g \otimes a) = \eta(g)a$ for $g \in F(A,B)$ and $a \in A$, i.e.*

$$F(A,B) := (A,B)^c/\mathrm{Ann}(A,B_\omega) \cong A' \cap \mathcal{M}(D_{A,B})\,.$$

(6) *$(A,B)^c$ is unital, if and only if, B_ω is unital, if and only if, B is unital.*
(7) $\mathrm{Ann}(A,B_\omega)=\{0\}$, *if and only if, B is unital and $1_B\in A$.*
(8) *Suppose that $d\in A_+$ is a full positive contraction, and let $E:=\overline{dAd}$. Then the natural *-morphism*

$$c\in A'\cap\mathcal{M}(D_{A,B})\mapsto c\in E'\cap\mathcal{M}(D_{E,B})$$

is bijective and defines an isomorphism ψ from $F(A,B)$ onto $F(E,B)$ with

$$\rho_{A,B}(c\otimes a)=\rho_{E,B}(\psi(c)\otimes a)$$

for $c\in F(A,B)$ and all $a\in E$.
(9) *If $C\subset B$ is a hereditary C^*-subalgebra with $A\subset C_\omega\subset B_\omega$, then $(A,B)^c=(A,C)^c+\mathrm{Ann}(A,B_\omega)$ and $F(A,B)\cong F(A,C)$.*

The proof is given in Appendix B. The only non-trivial parts are (4) and (8). Part (9) and the proof of part (8) show that

$$F(A_1,B_1)\cong F(A_2,B_2)$$

if the pairs (A_1,D_{A_1,B_1}) and (A_2,D_{A_2,B_2}) are Morita equivalent, and A_1, A_2 are both σ-unital.

The proofs of parts (5) and (8) use part (4) and a lemma on Morita equivalence of non-degenerate pairs $A_j\subset D_j$ (cf. Lemma B.1). Part (7) follows from part (6) and Remark 2.7.

Corollary 1.10 *Suppose that A is σ-unital. Then*

(1) $\mathrm{Ann}(A)$ *is an ideal of A^c and $F(A)$ is unital.*
(2) *A is unital, if and only if,* $\mathrm{Ann}(A)=\{0\}$, *if and only if, A^c is unital.*
(3) *$F(E)\cong F(A)$ if E is σ-unital and Morita equivalent to A.*
(4) *If $b\in A_+$ is a full positive element of A and $E:=\overline{bAb}$, $D_E:=\overline{bA_\omega b}\subset B_\omega$ then $\rho_A\colon F(A)\otimes^{\max}A\to A_\omega$ induces an* isomorphism *ψ from $F(A)$ onto $E'\cap\mathcal{M}(D_E)$ with $\psi(d)c=\rho_A(d\otimes c)=fb$ for $n\in\mathbb{N}$, $c\in E$ and $d\in F(A)$, where $f\in A^c$ is any element with $f+\mathrm{Ann}(A)=c$.*
(5) *Let $f\in A^c$ and $b\in A_+$ a* full *element of A, then $\|d\|=\lim_{n\to\infty}\|b^{1/n}f\|$ for every $d\in F(A)$ and $f\in A^c$ with $d=f+\mathrm{Ann}(A)$.*
(6) *$F(A)=F(A,A+\mathbb{C}\cdot 1)=F(A,\mathcal{M}(A))$ and*

$$F(A+\mathbb{C}\cdot 1)=(A+\mathbb{C}\cdot 1)^c=A^c+\mathbb{C}\cdot 1\subset(A+\mathbb{C}\cdot 1)_\omega\cong A_\omega+\mathbb{C}\cdot 1\,.$$

Part (6) follows from part (9) of 1.9.

Remark 1.11 *If $A\subset B_\omega$ and A is σ–unital, then*

$$F(A,B)\cong A'\cap\mathcal{M}(D_{A,B})=\mathcal{M}(A)'\cap\mathcal{M}(D_{A,B})\,.$$

and

$$\mathcal{Z}(\mathcal{M}(A))\cup\mathcal{Z}(\mathcal{M}(D_{A,B})\subset\mathcal{Z}(F(A,B))\subset F(A,B)\,.$$

□

Proposition 1.12 *Suppose that B is separable, A is a separable C^*-subalgebra of B_ω, and D is a separable C^*-subalgebra of $F(B)$ with $1_{F(B)} \in D$,*

*(1) There is a *-morphism $\psi\colon C_0((0,1],D) \to (A,B)^c$ with $\psi(f_0 \otimes 1)b = b$ for all $b \in A$, i.e. $\psi(C_0((0,1),D)) \subset \mathrm{Ann}(A,B_\omega)$ and*

$$[\psi]\colon\ d \in D \to \psi(f_0 \otimes d) + \mathrm{Ann}(A,B_\omega) \in F(A,B)$$

*is a unital *-monomorphism from D into $F(A,B)$.*

(2) If in addition, $B \subset A$, then $\psi\colon C_0((0,1],D) \to (A,B)^c$ in (2) can be found such that, moreover, $\psi(C_0((0,1),D)) = \psi(C_0((0,1],D)) \cap \mathrm{Ann}(B)$, i.e. that $[\psi](D)$ has trivial intersection with the image of $(A,B)^c \cap \mathrm{Ann}(B)$ in $F(A,B)$.

By induction, part (2) of Proposition 1.12 implies:

Corollary 1.13 *If A is separable and $C, B_1, B_2, \ldots$ are separable unital C^*-subalgebras of $F(A)$, then there is a unital a *-morphism*

$$\psi\colon\ C \otimes^{\max} B_1 \otimes^{\max} B_2 \otimes^{\max} \ldots \to F(A)$$

*with $\psi(c \otimes 1 \otimes 1 \otimes \cdots) = c$ for $c \in C$, such that the *-morphisms*

$$b \in B_n \mapsto \psi(1 \otimes \cdots 1 \otimes b \otimes 1 \otimes \cdots) \in F(A)$$

are faithful.

The stable invariant $F(A)$ has the following local continuity property:

Proposition 1.14 *Let $A_1 \subset A_2 \subset \ldots A$ C^*-subalgebras such that $\bigcup_n A_n$ is dense in A and A is separable. Then for every separable unital C^*-subalgebra B of the ultrapower*

$$\prod_\omega \{F(A_1), F(A_2), \ldots\} := \ell_\infty\{F(A_1), F(A_2), \ldots\}/c_\omega\{F(A_1), F(A_2), \ldots\}$$

*there is a unital *-morphism from B into $F(A)$.*

*In particular, for every simple separable unital *-subalgebra D of $F(A)_\omega$, there is a copy of D unitally contained in $F(A)$.*

See Appendix B for the proof.

If J is a closed ideal of B, let $\eta_J\colon B \to \mathcal{M}(J)$ and $\pi_J\colon B \to B/J$ the natural *-morphisms. We denote by $\eta := (\eta_J)_\omega\colon B_\omega \to \mathcal{M}(J)_\omega$ and $\pi := (\pi_J)_\omega\colon B_\omega \to (B/J)_\omega$ the ultrapowers of η_J and π_J.

Recall that $(X,J)^c := X' \cap J_\omega = (X, \mathcal{M}(J))^c \cap J_\omega$ for C^*-subalgebras $X \subset \mathcal{M}(J)_\omega$.

Remark 1.15 *Suppose that J is a closed ideal of B and that $A \subset B_\omega$ is a separable C^*-subalgebra. Let $\eta := (\eta_J)_\omega$ and $\pi := (\pi_J)_\omega$ as above.*

(1)

$$\operatorname{Ann}(A, J_\omega) := \operatorname{Ann}(A, B_\omega) \cap J_\omega = \operatorname{Ann}(\eta(A), J_\omega)\,,$$
$$(A, J)^c := A' \cap J_\omega = (A, B)^c \cap J_\omega = \eta(A)' \cap J_\omega = (\eta(A), J)^c$$
$$(\pi_J)_\omega(\operatorname{Ann}(A, B_\omega)) = \operatorname{Ann}(\pi(A), (B/J)_\omega)\,.$$
and
$$(\pi_J)_\omega((A, B)^c) = (\pi(A), B/J)^c\,.$$

In particular, $F(A, J) := (\eta(A), J)^c/\operatorname{Ann}(\eta(A), J_\omega)$, *is isomorphic to the ideal* $(((A, B)^c \cap J_\omega) + \operatorname{Ann}(A, B_\omega))/\operatorname{Ann}(A, B_\omega)$ *of* $F(A, B) = (A, B)^c/ \operatorname{Ann}(A, B_\omega)$.

(2) The sequences

$$0 \to (A, J)^c \to (A, B)^c \to (\pi(A), B/J)^c \to 0$$
$$0 \to \operatorname{Ann}(A, J_\omega) \to (A, J)^c \to F(A, J) \to 0$$
$$0 \to F(A, J) \to F(A, B) \to F(\pi(A), B/J) \to 0.$$

are short-exact and strongly locally semisplit in the sense of Definition 1.5.

(3) If J *is a closed ideal of* $A = B$*, then the natural *-morphism* $F(A) \to F(A/J)$ *is an epimorphism with kernel* $F(A, J)$*, if* J *is a closed ideal of* A*, and there is a unital *-morphism* $F(A) \to F(J) \cong (J, A)^c/\operatorname{Ann}(J, A_\omega)$ *with kernel* $(\operatorname{Ann}(J, A_\omega) \cap A^c)/\operatorname{Ann}(A)$.

One gets the first two lines of part (1) by straight calculations. Then parts (1)-(3) follow from Proposition 1.6, Corollary 1.7 and Proposition 1.9(9).

Corollary 1.16 *Suppose that* $J \triangleleft B$ *is an* essential *ideal of* B *and that* A *is a separable* C^**-subalgebra of* B_ω*. Then* $F(A, J)$ *is an essential ideal of* $F(A, B)$.

Proposition 1.17 *Suppose that* B *is a separable unital* C^**-algebra,* J *a closed ideal of* B*, and* $1_B \in A \subset B_\omega$ *is a separable* C^**-subalgebra. If* D_1 *is a unital separable* C^**-subalgebra of* $F((\pi_J)_\omega(A), B/J) = ((\pi_J)_\omega(A), B/J)^c$ *and* D_0 *is a unital separable* C^**-subalgebra of* $F(J)$ *then there is a unital *-morphism* $h\colon\ \mathcal{E}(D_0, D_1) \to F(A, B) = (A, B)^c$.

Here

$$\mathcal{E}(D_0, D_1) := \{f \in C([0, 1],\ D_0 \otimes^{\max} D_1)\,;\ f(0) \in D_0 \otimes 1,\ f(1) \in 1 \otimes D_1\,\}.$$

The proof in Appendix B gives h with the additional property $\pi(h(f)) = f(1)$ for $f \in \mathcal{E}(D_0, D_1)$ and the natural *-morphisms $\pi\colon\ F(A, B) \to F((\pi_J)_\omega(A), B/J)$.

Note that $\mathcal{E}(D_0, D_2)$ is unitally contained in $\mathcal{E}(D_0, D_1)$ if $D_2 \subset D_1$. There is a unital *-morphism $D_1 \to \mathcal{E}(D_1, \mathcal{O}_2)$ if D_1 is simple and nuclear, because then $D_1 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ and all unital *-monomorphisms of separable unital exact

C^*-algebras into $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \ldots$ are homotopic (by basics of classification). We apply Proposition 1.17 to the extensions

$$0 \to J \otimes \mathcal{K} \to (A \otimes \mathcal{K}) + \mathcal{O}_2 \to ((A/J) \otimes \mathcal{K}) + \mathcal{O}_2 \to 0$$

and

$$0 \to (A/J) \otimes \mathcal{K} \to ((A/J) \otimes \mathcal{K}) + \mathcal{O}_2 \to \mathcal{O}_2 \to 0\,,$$

where $\mathcal{O}_2 \subset \mathcal{M}(\mathcal{K})$ unitally, and we use the unital *-morphism from $F((A \otimes \mathcal{K}) + \mathcal{O}_2)$ into $F(A \otimes \mathcal{K}) \cong F(A)$. We obtain finally:

Corollary 1.18 *Let A a separable C^*-algebra and J a closed ideal of A. If D_1 is a unital separable C^*-subalgebra of $F(A/J)$ and D_0 is a unital separable C^*-subalgebra of $F(J)$ then there exists a unital *-morphism $h\colon \mathcal{E}(D_0, \mathcal{E}(D_1, \mathcal{O}_2)) \to F(A)$.*

*If, moreover, D1 is simple and nuclear, then there exists a unital *-morphism from $\mathcal{E}(D_0, D_1)$ into $F(A)$.*

2 The Case of Simple $F(A)$

We study here some particular ideals of $F(A)$ for separable A. The less trivial basic facts are given in Lemmas 2.8, 2.9 and 2.11(3,4). We apply them in the particular case where A is simple, to get the main result of this section: Theorem 2.12. Then we have a look to non-separable A, and we pose some related problems.

The following Lemma seems to be wrong for *non-separable* A, because it might be that $F(\mathcal{L}(\ell_2)) = \mathcal{L}(\ell_2)^c \cong \mathbb{C}$, cf. Question 2.22.

Lemma 2.1 *Suppose that A is separable. If $F(A)$ is simple, then A is simple.*

More precisely, if J is a non-trivial closed ideal of A, then

(1) J_ω is a closed ideal of A_ω with $A \cap J_\omega = J$, and
(2) $F(A, J) = (A^c \cap J_\omega)/(\mathrm{Ann}(A) \cap J_\omega)$ is a non-trivial closed ideal of $F(A)$.

$F(A, J)$ is an essential ideal of $F(A)$ if J is essential, cf. Corollary 1.16.

If even A^c is simple, then A is simple *and* unital, because then $F(A) = A^c/\{0\}$ is simple, and $\mathrm{Ann}(A) = \{0\}$ implies that A is unital (for σ-unital A, cf. Corollary 1.10).

Proof. (1): It is clear that J_ω is a closed ideal of A_ω. If $a \in A \cap J_\omega$ then there is a bounded sequence $b_1, b_2, \ldots \in J$ with $\lim_\omega \|a - b_n\| = 0$. Thus, there is a sub-sequence $c_k := b_{n_k} \in J$ with $\lim_{k\to\infty} c_k = a$, i.e. $a \in J$.

(2): $A^c \cap J_\omega$ is a closed ideal of A^c by (1).

Since A is separable, J is separable and contains a strictly positive contraction $b \in J_+$ for J, moreover, there are $b_1, b_2, \ldots \in C^*(b)_+$ with $\|b_n\| = 1$,

$b_nb_{n+1} = b_n$, $\|b - b_nb\| < 1/n$ and $\lim_{n\to\infty} \|b_na - ab_n\| = 0$ for all $a \in A$ (cf. the proof of [29, thm. 3.12.14]).

Thus $c := \pi_\omega(b_1, b_2, \ldots)$ is in $A^c \cap J_\omega$ and $cb = b \neq 0$. Thus $c \notin \mathrm{Ann}(A)$, i.e. $A^c \cap J_\omega \not\subset \mathrm{Ann}(A)$ and $F(A,J) = A^c \cap J_\omega/(\mathrm{Ann}(A) \cap J_\omega)$ is a non-zero closed ideal of $F(A)$.

Let a_0 is a strictly positive contraction in A_+. $F(A,J) \neq F(A)$, because otherwise there is a positive contraction $e \in A^c \cap J_\omega$ with $e + \mathrm{Ann}(A) = 1 \in F(A)$ and $a_0 = \rho(1 \otimes a_0) = ea_0 \in J_\omega$, i.e. $a_0 \in J$ by (1), which contradicts the non-triviality of J. □

A modification of the proof of Lemma 2.1(2) shows:

Remark 2.2 *If* $\{0\} \neq J \neq A$, *then* $I := A^c \cap \mathrm{Ann}(J, J_\omega)$ *is a closed ideal of* A^c *that is not contained in* $\mathrm{Ann}(A)$.
Note that $b \in I/(I \cap \mathrm{Ann}(A)) \subset F(A)$ *if and only if* $\rho_A(b \otimes J) = 0$.

Lemma 2.3 *If A is antiliminal then for every positive* $b \in A_\omega$ *with* $\|b\| = 1$ *there exists a *-monomorphism ψ from* $C_0((0,1],\mathcal{K})$ *into* A_ω *with* $b\psi(c) = \psi(c)$ *for every* $c \in C_0((0,1],\mathcal{K})$.

Recall that "antiliminal" means that $\{0\}$ is the only Abelian hereditary C^*-subalgebra of A.

Proof. Let $(b_1, b_2, \ldots) \in \ell_\infty(A)_+$ a representing sequence for b with $\|b_n\| = 1$, let $d_n := (b_n - (n-1)/n)_+ \neq 0$ and let D_n denote the closure of d_nAd_n. Then $bc = c$ for all elements c in $\prod_\omega \{D_n\,;\, n \in \mathbb{N}\} \subset A_\omega$.

Since $C_0((0,1],\mathcal{K}) \subset \prod_\omega \{C_0((0,1], M_n)\,;\, n \in \mathbb{N}\}$, is suffices to find faithful *-morphisms $\psi_n \colon C_0((0,1], M_n) \to D_n$. By the Glimm halving lemma (cf. [29, lem. 6.7.1]) there is a non-zero *-morphism $h_n \colon C_0((0,1], M_n) \to D_n$ because D_n is antiliminal as well (because it is a non-zero hereditary C^*-subalgebra of A). Let E_n the hereditary C^*-subalgebra of $D_n \subset A$ generated by $h_n(f_0 \otimes e_{1,1})$. If M is a maximal Abelian C^*-subalgebra of E_n with $h_n(f_0 \otimes e_{1,1}) \in M$, then M can not contain a minimal idempotent p, because otherwise $pAp = \mathbb{C}p$ which contradicts that A is antiliminal. It follows that h_n can be replaced by a *-*mono*morphism $\psi_n \colon C_0((0,1], M_n) \to D_n$. □

Remark 2.4 *Let A a σ-unital C^*-algebra.*
The closed ideal J_A of A_ω generated by A is simple, if and only if, either A is simple and purely infinite or A is isomorphic to the compact operators $\mathcal{K}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$.
If $A \not\cong \mathcal{K}(\mathcal{H})$ and J_A is simple, then A_ω and A are simple and purely infinite.
If $A \cong \mathcal{K}(\mathcal{H})$, *then* $J_A \cong \mathcal{K}(\mathcal{H}_\omega)$.
(If $A \cong \mathcal{K}(\mathcal{H})$ and is $\sigma-unital$, then $\mathcal{H}$ is separable, and $\mathrm{Dim}(\mathcal{H}) = \infty$ *if and only if* $J_A \neq A_\omega$.*)*

Proof. If $A \cong \mathcal{K}(\mathcal{H})$ then $J_A \cong \mathcal{K}(\mathcal{H}_\omega)$, because there is a natural *-monomorphism λ from $\mathcal{K}(\mathcal{H})_\omega$ into $\mathcal{L}(\mathcal{H}_\omega)$ that satisfies

$$\lambda\left(\left(\langle ., x_n\rangle y_n\right)_\omega\right) = \langle ., x_\omega\rangle y_\omega\,.$$

$\lambda(\mathcal{K}(\mathcal{H})) = P\mathcal{K}(\mathcal{H}_\omega)P$ is the hereditary C^*-subalgebra of $\mathcal{K}(\mathcal{H}_\omega)$ that is defined by the orthogonal projection P from $\mathcal{H}_\omega$ onto $\mathcal{H} \subset \mathcal{H}_\omega$. Since $\lambda(\mathcal{K}(\ell_2)_\omega) \not\subset \mathcal{K}((\ell_2)_\omega)$, it holds $A_\omega = \mathcal{K}(\mathcal{H})_\omega \subset \mathcal{K}(\mathcal{H}_\omega)$ if and only if $\mathcal{H}_\omega = \mathcal{H}$ if and only if and $\mathrm{Dim}(\mathcal{H}) = n < \infty$.

It is easy to see (with help of representing sequences in case $\|b\| = \|c\| = 1$) that for every $b, c \in (A_\omega)_+$ there is a contraction $d \in (A_\omega)_+$ with $\|c\|d^*bd = \|b\|c$ if A is simple and purely infinite. Thus A_ω is simple and purely infinite, and $J_A = A_\omega$.

Conversely, suppose that J_A is simple. This implies that A must be simple, because otherwise $J_A \cap I_\omega \supset I$ is a non-trivial closed ideal of J_A if I is a non-trivial closed ideal of A. Suppose that $A \ncong \mathcal{K}(\mathcal{H})$ (for any Hilbert space $\mathcal{H}$), i.e. that A is antiliminal. Let $b, c \in (J_A)_+$ with $\|b\| = \|c\|$. Since A is antiliminal, by Lemma 2.3 there exists a *-monomorphism $\psi\colon C_0((0,1],\mathcal{K}) \hookrightarrow A_\omega$ with $b\psi(f) = \psi(f)$ for every $f \in C_0((0,1],\mathcal{K})$. Let D denote the hereditary C^*-subalgebra of A_ω generated by the image of ψ. D is non-zero, stable and satisfies $bg = g = gb$ for all $g \in D$. In particular, $D \subset J_A$. Since J_A is simple and D is stable, there is $d \in J_A$ with $d^*d = c$ and $dd^* \in D$. Thus $d^*bd = d^*d = c$. It follows that A is purely infinite, because we can take $b, c \in A$ and find a representing sequence $(d_1, d_2, \ldots) \in \ell_\infty(A)$ for d with $d^*bd = c$ in A_ω. □

Lemma 2.5 *Suppose that B is a separable C^*-subalgebra of A_ω.*

If λ is a pure state on B, then there exists a sequence of pure states $\mu_1, \mu_2, \ldots$ on A such that λ is the restriction of the state $\mu_\omega\colon A_\omega \to \mathbb{C} \cong \mathbb{C}_\omega$ to B.

If $(\mu_1, \mu_2, \ldots)$ is any sequence of pure states on A, then there are positive contractions $g_n \in A_+$ such that $\mu_n(g_n) = 1$, and $gbg = \mu_\omega(b)g^2$ for all $b \in B$, where $g := \pi_\omega(g_1, g_2, \ldots)$.

Note that $\|g\| = 1$.

Proof. If C is a C^*-algebra and λ is a pure state on C, then for every *separable* C^*-subalgebra $B \subset C$ there is $c \in C_+$ with $\lambda(c) = \|c\| = 1$ and $\lim_{n\to\infty} \|c^n b c^n - \lambda(b)c^{2n}\| = 0$ for every $b \in B$ (cf. [6, lem. 2.14]).

Clearly, in the case $B = C$, the limes property of $c \in B$ implies that $\nu = \lambda$ for all $\nu \in B^*$ with $\nu(c) = \|\nu\| = 1$. (In fact, the latter property of c equivalent to the limes property of c.)

We find a sequence $c_1, c_2, \ldots \in A_+$ with $\|c_n\| = 1$ and $\pi_\omega(c_1, c_2, \ldots) = c$. Let $\mu_1, \mu_2, \ldots$ pure states on A with $\mu_n(c_n) = 1$. Then $\mu_\omega(c) = 1 = \|\mu_\omega\|$. Thus $\mu_\omega | B = \mu$.

Suppose that $(\mu_1, \mu_2, \ldots)$ is any sequence of pure states on A. Let $b^{(1)}, b^{(2)}$, $\ldots \in B$ a dense sequence in the unit ball of B. There are representing sequences $s_k = (b_1^{(k)}, b_2^{(k)}, \ldots) \in \ell_\infty(A)$ with $\|b_n^{(k)}\| \leq 1$ and $\pi_\omega(s_k) = b^{(k)}$. By the above mentioned result of [6, lem. 2.14], there are $c_n \in A_+$ and $p_n \in \mathbb{N}$ with $\|c_n\| = 1 = \mu_n(c_n)$ and $\|c_n^{p_n} b_n^{(k)} c_n^{p_n} - \mu_n(b_n^{(k)}) c_n^{2p_n}\| < 2^{-n}$ for $k \leq n = 1, 2, \ldots$.

Let $g_n := c_n^{p_n}$ for $n \in \mathbb{N}$. Then $g := \pi_\omega(g_1, g_2, \ldots) \in A_\omega$ satisfies $0 \leq g$, $\|g\| = 1 = \mu_\omega(g)$ and $gb^{(k)}g = \mu_\omega(b^{(k)})g^2$. □

Remark 2.6 *Let $g \in A_\omega$ with $0 \leq g$, $\|g\| = 1$ and $gbg = \mu(b)g^2$ for $b \in B \subset A_\omega$, then $g \in (B, A)^c$ if and only if μ is a character on B. (Left to the reader.)*

Remark 2.7 *Lemma 2.5 implies that*

$$\mathrm{Ann}(\mathrm{Ann}(B, A_\omega), A_\omega) = D_{B,A} := \overline{b_0 A_\omega b_0}$$

if B is σ-unital and $b_0 \in B$ is a strictly positive contraction for B.

Proof. If $a \in (A_\omega)_+$ is *not* in $D_{B,A}$, then $\inf_n \|(1 - b_0^{1/n})a(1 - b_0^{1/n})\| > 0$. Thus, there is a pure state μ on $C^*(b_0, a)$ with $\mu(b_0) = 0$ and $\mu(a) > 0$. By Lemma 2.5 there exists $g \in (A_\omega)_+$ with $\|g\| = 1$ and $gcg = \mu(c)g^2$ for $c \in C^*(b_0, a)$. Hence, $g \in \mathrm{Ann}(B, A_\omega)$ and $ag = g \neq 0$. □

The *socle* of a C^*-algebra A is the (algebraic) ideal generated by the projections $p \in A$ with $pAp = \mathbb{C} \cdot p$. If A is simple, then $\mathrm{socle}(A) \neq \{0\}$ if and only if $A \cong \mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Lemma 2.8 $\mathrm{socle}(F(A)) = \{0\}$ *if A is separable and* $\mathrm{socle}(A) = \{0\}$.

Proof. Let $p^2 = p^* = p \in F(A)$ a non-zero projection. We show that $pF(A)p \neq \mathbb{C} \cdot p$ if $\mathrm{socle}(A) = \{0\}$. The idea of the proof goes as follows: Let $s \in A^c$ a self-adjoint contraction with $s + \mathrm{Ann}(A) = 1 - 2p$, and $d := s_+$, $q := s_-$. We show below that there exist positive contractions $g, h \in A_\omega$ such that $dg = dh = hg = 0$, $gAh = \{0\}$, $Ah \neq \{0\}$ and $Ag \neq \{0\}$.

If we have found $g, h \in (A_\omega)_+$ with this properties, then Proposition 1.3 (with $J = A = B$, $\mathcal{V} = CP(A, A)$ and a, b, c replaced by d, g, h) yields: There are positive contractions $e, f \in A^c$ with $de = df = ef = 0$, $eg = g$, $fh = h$. It follows that $e' := e + \mathrm{Ann}(A)$ and $f' := f + \mathrm{Ann}(A)$ satisfy $e'p = e'$, $f'p = f'$ and $e'f' = 0$ in $F(A)$. Since $Ag \neq \{0\}$ and $Ah \neq \{0\}$, we get $e, f \notin \mathrm{Ann}(A)$ and $e' \neq 0 \neq f'$. Hence, $pF(A)p \neq \mathbb{C} \cdot p$.

The rest of the proof is concerned with the construction of the positive contractions $g, h \in A_\omega$ (as stipulated):

Step 1 (Construction of $a_0 \in A$ $E \subset A_\omega$, μ, ν): Consider the *-morphism $\psi\colon\ a \in A \mapsto \rho(p \otimes a) = qa \in A_\omega$. ψ is non-zero because $p \neq 0$ and $\rho((\cdot) \otimes A)$ is separating for $F(A)$. Let J denote the kernel of ψ, and let $a_1 \in J_+$ and $a_2 \in (A/J)_+$ strictly positive contractions with $\|a_1\| = \|a_2\| = 1$. There is a positive contraction $a_3 \in A_+$ with $a_3 + J = a_2$. $a_0 := (1 - a_1)^{1/2} a_3 (1 - a_1)^{1/2} + a_1$ is a strictly positive contraction of A with

$$\|qa_0\| = \|\rho(p \otimes a_0)\| = \|a_0 + J\| = \|a_2\| = 1.$$

Since q and a_0 are commuting positive contractions with $\|qa_0\| = 0$ there is a character χ on $C^*(a_0, q)$ with $\chi(qa_0) = \chi(q) = \chi(a_0) = 1$. By Lemma 2.5 there is a sequence $(\mu_1, \mu_2, \ldots)$ of pure states on A such that χ is the restriction of $\mu := (\mu_1, \mu_2, \ldots)_\omega$ to $C^*(a_0, q)$. The state $\mu\colon A_\omega \to \mathbb{C}$ is supported on the closure E of $a_0 q A_\omega a_0 q$. In particular $\mu(\mathrm{Ann}(E, A_\omega)) = \{0\}$. It follows $\mu(d) = 0$ because $dq = 0$.

E is contained in the closure D_A of $a_0 A_\omega a_0$. Thus, for every non-zero element $y \in E_+$ holds $Ay \neq \{0\}$.

Let $G = \{u_1 = 1, u_2, \ldots\}$ a countable dense subgroup of the unitary group of $\widetilde{A} = A + \mathbb{C} \cdot 1$, and $\nu(b) := \sum_{n=1}^{\infty} 2^{-n} \mu(u_n^* b u_n)$ for $b \in E$. Since a_0 is strictly positive in A and q commutes with A, we get that $AE + EA \subset E$, ν is a state on E with $\mu \leq 2\nu$. Clearly $b \in L_\nu := \{x \in E\,;\nu(x^*x) = 0\}$ if and only if $bu \in L_\mu := \{x \in E\,; \mu(x^*x) = 0\}$ for all $u \in G$. Thus, $L_\nu G \subset L_\nu$ and $L_\nu A \subset L_\nu$.

Step 2 (ν is not *faithful on E):* We find a representing sequence $(c_1, c_2, \ldots) \in \ell_\infty(A)_+$ with $\pi_\omega(c_1, c_2, \ldots) = a_0 q$ and $\|c_n\| = 1$. Let C_n a maximal Abelian C^*-subalgebra of $D_n := \overline{(c_n - 1/2)_+ A (c_n - 1/2)_+}$ that contains $(c_n - 1/2)_+$. C_n does not contains a *minimal idempotent* $r \neq 0$, because otherwise r must satisfy $rAr = \mathbb{C} \cdot r$, i.e. $r \in \mathrm{socle}(A) = \{0\}$. Hence, the primitive ideal space of C_n is a perfect locally compact metric space and there is $f_n \in (C_n)_+$ with $\mathrm{Spec}(f_n) = [0, 1]$, i.e. C_n contains a copy of $C_0((0, 1])$ up to isomorphisms.

The corresponding monomorphic image C of $C_0((0, 1])_\omega$ in A_ω satisfies $wb = bw = b$ for $b \in C$ and $w := 2(a_0 q - (a_0 q - 1/2)_+)$, thus $C \subset E$.

Let $x \in (0, 1)$ and $f_{x,n}(t)$ the continuous function in $t \in [0, 1]$ with $f_{x,n}(x) = 1$, $f_{x,n}(t) = 0$ for $t \in [0, x - \min(1/n, x)] \cup [x + \min(1/n, 1 - x), 1]$ and $f_{x,n}$ is linear on $[x - \min(1/n, x), x]$ and $[x, x + \min(1/n, 1 - x)]$.

$\delta_x := \pi_\omega(f_{x,1}, f_{x,2}, \ldots)$ is a positive contraction in $C_0((0, 1])_\omega \cong C$ with $\|\delta_x\| = 1$ and $\delta_x \delta_y = 0$ for $x \neq y$.

It follows that $E \supset C$ contains uncountably many pair-wise orthogonal non-zero positive contractions, because $\{\delta_x\}_{x \in (0,1)}$ is a family of pair-wise orthogonal positive elements in $C_0((0, 1])_\omega \cong C$ with $\|\delta_x\| = 1$. Hence ν can not be faithful on E, i.e. $D := L_\nu^* \cap L_\nu = L_\nu^* L_\nu$ is a non-zero hereditary C^*-subalgebra of E.

Step 3 (Construction of $g, h \in E_+$): Let $D := L_\nu^* \cap L_\nu$ and let $h \in D_+$ with $\|h\| = 1$. Then $dh = 0$, $Ah \neq 0$, $AD + DA \subset D$ and $\mu(a^* h^2 a) \leq 2\nu(a^* h^2 a) = 0$ for all $a \in A + \mathbb{C} \cdot 1$, because $L_\nu A \subset L_\nu \subset E \subset \mathrm{Ann}(d, A_\omega) \cap D_A$. By Lemma 2.5 there is $g \in (A_\omega)_+$ with $\|g\| = 1$ such that $gyg = \mu(y) g^2$ for all $y \in C^*(A, q, d, h)$, because $\mu_1, \mu_2, \ldots$ are pure states on A. It follows $gd^2 g = \mu(d^2) g^2 = 0$, $gh^2 g = \mu(h^2) g^2 = 0$, $ga^* h^2 ag = \mu(a^* h^2 a) g^2 = 0$, and $ga_0 g = \mu(a_0) g^2 = g^2 \neq 0$, i.e. $g, h \in A_\omega$ are as required. □

Lemma 2.9 *Suppose that A is separable.*

(1) $\mathrm{Prim}(A)$ *is quasi-compact, if and only if, for every non-invertible $e \in F(A)_+$, there exists non-zero $d \in F(A)_+$ with $de = 0$.*

(2) If $\mathrm{Prim}(A)$ *is quasi-compact, then every maximal family of mutually orthogonal positive contractions in $F(A)$ is either finite and has invertible sum or is not countable.*

Part (2) applies to *simple* C^*-algebras A, because $\mathrm{Prim}(A)$ is a singleton if and only if A is simple. The Bourbaki terminology "quasi-compact" is used for non-Hausdorff T_0 spaces.

Proof. (1): Recall that there is a one-to-one isomorphism from the lattice of closed ideals of A onto the lattice of open subsets of $\mathrm{Prim}(A)$. Since A is separable, $\mathrm{Prim}(A)$ is second countable. Thus, if $\mathrm{Prim}(A)$ is not quasi-compact, then there is an increasing sequence $J_1 \subset J_2 \subset \cdots$ of closed ideals of A with $J_n \neq J_{n+1}$ and $\bigcup_n J_n$ dense in A. For each $n \in \mathbb{N}$ there exists a positive contraction $c_n \in J_n$ with $\|c_n + J_{n-1}\| = 1$ such that $c_n + J_{n-1}$ is a strictly positive element of J_n/J_{n-1} (where we let $J_0 := \{0\}$). Then $a_n := \sum_{1\leq k\leq n} 2^{-k} c_k$ is a strictly positive contraction in J_n, and $b_0 := \sum_{1\leq n<\infty} 2^{-n} c_n$ is a strictly positive contraction in A (with norm $\geq 1/2$). Let $f_0 := 0$. By induction, we find positive contractions $f_n \in (A, J_n)^c = A' \cap (J_n)_\omega \subset A^c$ with $a_n f_n = a_n$ and $f_{n-1} f_n = f_{n-1}$ (cf. Remark 1.15(1)). Let $f := \sum_{1\leq n<\infty} 2^{-n} f_n \in (A^c)_+$ and let $e := f + \mathrm{Ann}(A) \in F(A)$. Then $fb_0 = b_0 f$ is positive, $a_n = f_n a_n \leq 2^n f_n b_0 \leq 4^n f b_0$ for $n \in \mathbb{N}$, and $\|f - f_{k+1} f\| \leq 2^{-k}$. If $d \in (A^c)_+$ satisfies $df \in \mathrm{Ann}(A)$, then $dfb_0 = 0$ and $da_n = 0$ for all $n \in \mathbb{N}$. Hence, $db_0 = 0$ and $d \in \mathrm{Ann}(A)$ whenever $d \in (A^c)_+$ and $df \in \mathrm{Ann}(A)$. Let $e := f + \mathrm{Ann}(A)$. Then $g = 0$, if $g = d + \mathrm{Ann}(A) \in F(A)_+$ with $d \in A^c_+$ and $ge = 0 \in F(A)$.

The image $F(A, J_n)$ of $(A, J_n)^c = (J_n)_\omega \cap A^c$ in $F(A)$ is a *non-trivial* ideal of $F(A)$ by Lemma 2.1(2), because J_n is a non-trivial ideal of A. Since $f_n \in (J_n)_\omega \cap A^c$, the element $e_n := f_n + \mathrm{Ann}(A) \in F(A)_+$ is not invertible in $F(A)$. It follows that $e := f + \mathrm{Ann}(A)$ is not invertible, because $\|e_{k+1} e - e\| \leq 2^{-k}$.

Conversely, suppose that $\mathrm{Prim}(A)$ is quasi-compact and that $e \in F(A)_+$ is not invertible. We can suppose that $\|e\| = 1$. Then there is a contraction $f \in A^c_+$ with $e = f + \mathrm{Ann}(A)$. Let $a_0 \in A_+$ a strictly positive contraction with $\|a_0\| = 1$, and let J_n denote the closure of $\mathrm{span}(A(a_0 - 1/n)_+ A)$. Then J_n is an increasing sequence of closed ideals of A with $\bigcup_n J_n$ dense in A, i.e. the corresponding increasing sequence of open subsets of $\mathrm{Prim}(A)$ covers $\mathrm{Prim}(A)$. Since $\mathrm{Prim}(A)$ is quasi-compact, there is $n \in \mathbb{N}$ such that $J_n = A$, i.e. that $b := (a_0 - 1/n)_+$ is a *full* positive contraction in A_+. Let $c := 2n((a_0 - 1/(2n))_+ - (a_0 - 1/n)_+)$, then $bc = b$ Note that $C^*(b, c, f) \subset A_\omega$ is an Abelian C^*-algebra.

By Corollary 1.10, $\rho\colon F(A) \otimes^{\max} A \to A_\omega$ induces an *isomorphism* ψ from $F(A)$ onto $(bAb)' \cap \mathcal{M}(D_b)$ with $\psi(e)(b^n) = \rho(e \otimes b^n) = fb^n$ for $n \in \mathbb{N}$, where $D_b := \overline{bA_\omega b}$ denotes the hereditary C^*-subalgebra of A_ω generated by b.

$(b - nfb)_+ \neq 0$ for each $n \in \mathbb{N}$, because, otherwise, there is $n \in \mathbb{N}$ with $b^2 \leq n^2 (fb)^2$ and

$$\|\psi(e)(bxb)\|^2 = \|\rho(e \otimes bxb)\|^2 = \|bx^* b f^2 bxb\| \geq n^{-2} \|bxb\|^2$$

for all $x \in A_\omega$, which contradicts that $\psi(e) \in \mathcal{M}(D_b)_+$ is *not* invertible. Thus, for every $n \in \mathbb{N}$ there exists a character χ_n on $C^*(b, c, f)$ with $\chi_n(b - nfb) > 0$, i.e. $\chi_n(b) > 0$ and $\chi_n(f) < 1/n$. Since $cb = b$, it follows $\chi_n(c) = 1$. The set of characters χ on $C^*(b, c, f)$ with $\chi(c) = 1$ is compact in the space of characters on $C^*(b,c,f)$. Let χ a character on $C^*(b,c,f)$ that is a cluster point of the sequence $\chi_1, \chi_2, \ldots$, then $\chi(f) = 0$ and $\chi(c) = 1$.

By Lemma 2.5 there exists $g \in (A_\omega)_+$ with $\|g\| = 1$, $gfg = \chi(f)g^2 = 0$ and $gcg = \chi(c)g^2 = g^2$. Then $fg = 0$ and $cg = g$. By Proposition 1.3 (with $J = A = B$, $\mathcal{V} = CP(A,A)$, and $f, 0, g$ in place of a, b, c) there there are positive contractions $h, k \in A^c$ with $kf = f$, $hg = g$ and $hk = 0$. Since $hc^2 \geq cgc \neq 0$ and $hf = hkf$, we get $h \notin \mathrm{Ann}(A)$ and $hf = 0$. Let $d := h + \mathrm{Ann}(A) \in F(A)_+$, then $d \neq 0$ and $de = 0$.

(2): If $e_1, e_2, \ldots \in F(A)$ is a sequence of pairwise orthogonal positive contractions, and $e := \sum 2^{-n} e_n$. If e is invertible, then $e_n = 0$ for $n \leq n_0$. If e is not invertible, then there exists non-zero $d \in (F(A))_+$ with $ed = 0$ by (1). Thus $e_n d = 0$ for all $n \in \mathbb{N}$. □

The following proposition characterizes $A \cong \mathcal{K}(\mathcal{H})$ by properties of $F(A)$ if A is simple and separable.

Proposition 2.10 *Suppose that A is separable and simple. The following are equivalent:*

(1) $A \otimes \mathcal{K} \cong \mathcal{K}$.
(2) $F(A) \cong \mathbb{C}$.
(3) $\mathrm{socle}(F(A)) \neq \{0\}$.
(4) $F(A)$ *is separable.*
(5) $F(A)$ *is simple and stably finite.*
(6) $F(A)$ *has a faithful finite quasi-trace.*
(7) Every commutative C^*-subalgebra of $F(A)$ admits a faithful state.
(8) Every family of mutually orthogonal positive contractions is at most countable.

Note that (2) also implies that A is simple (for separable A, by Lemma 2.1), thus $F(A) \cong \mathbb{C}$ *if and only if* $A \otimes \mathcal{K} \cong \mathcal{K}$ (for separable A). Clearly one can restrict in (7) to maximal Abelian C^*-subalgebras.

Proof. The implications (2) ⇒ (4) ⇒ (7) ⇒ (8), (2) ⇒ (5), (6) ⇒ (7), and (2) ⇒ (3) are obvious.

(5) ⇒ (6) follows from [9], because $F(A)$ is unital and finite dimension-functions on simple unital C^*-algebras B integrate to faithful finite quasi-traces on B.

(1)⇒(2): $F(A) \cong F(A \otimes \mathcal{K})$ and $F(\mathcal{K}) \cong F(\mathbb{C}) = \mathbb{C}$ by Corollary 1.10.

(3)⇒(1): $\mathrm{socle}(A) \neq \{0\}$ follows from (3) by Lemma 2.8. Thus $A \cong \mathcal{K}(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$, i.e. $A \otimes \mathcal{K} \cong \mathcal{K}$.

(8)⇒(3): Let $C \subset F(A)$ a maximal commutative C^*-subalgebra of $F(A)$. Every family $X \subset C_+$ of mutually orthogonal positive contractions in C is contained in a maximal family $Y \subset F(A)_+$ of mutually orthogonal positive contractions in $F(A)$. By (8) and Lemma 2.9(2), $Y \supset X$ is finite. Thus, the primitive ideal space $\widehat{C}$ of C can contain only a finite number of points, i.e. C is of finite dimension. If $p \neq 0$ is a minimal idempotent of C, then $p^* = p = p^2$ and $pF(A)p \cong \mathbb{C} \cdot p$ (by maximality of C). □

We call a completely positive map $T\colon A \to A_\omega$ *ω-nuclear* if there is a bounded sequence of nuclear c.p. maps $T_n\colon A \to A$ such that $T = T_\omega|A$. Here $T_\omega\colon A_\omega \to A_\omega$ means the ultrapower $(T_1, T_2, \ldots)_\omega$ of the bounded sequence of maps $(T_n\colon A \to A)_n$ given by $T_\omega(\pi_\omega(a_1, a_2, \dots)) := \pi_\omega(T_1(a_1), T_2(a_2), \ldots)$.

Lemma 2.11 *Suppose that A is a separable C^*-algebra.*

(1) The set $\mathcal{C}_{\omega\mathrm{nuc}}$ of ω-nuclear completely positive maps $V\colon A \to A_\omega$ is a point-norm closed (matricially) operator-convex cone (cf. Definition 1.2).
(2) Let κ denote the set of positive elements $b \in F(A)_+$ with the property that the c.p. map

$$a \in A \mapsto \rho_A(b \otimes a) \in A_\omega$$

is ω-nuclear. Then κ is the positive part of a closed ideal J_{nuc} of $F(A)$.
(3) J_{nuc} is an essential ideal of $F(A)$. In particular, J_{nuc} is non-zero for every separable C^*-algebra $A \neq \{0\}$.
(4) $J_{\mathrm{nuc}} = F(A)$ if and only if A is nuclear.

Proof. (1): Obviously, $\mathcal{V}_\omega \subset CP(A_\omega, B_\omega)$ is operator-convex in the sense of Definition 1.2, if $\mathcal{V} \subset CP(A, B)$ is an operator-convex cone and if $\mathcal{V}_\omega$ denotes the set of ultapowers of bounded sequences $T_1, T_2, \dots$ in $\mathcal{V}$. That is, $\mathcal{V}_\omega$ is a convex subcone of $CP(A_\omega, B_\omega)$ and $bT(a^*(\cdot)a)b^* \in \mathcal{V}_\omega$ for $T = (T_1, T_2, \ldots)_\omega \in \mathcal{V}_\omega$ and rows $a \in M_{1,n}(A_\omega)$, $b \in M_{1,n}(B_\omega)$. We get an operator-convex subcone $\mathcal{V}_\omega|A$ of $CP(A, B_\omega)$, if we restrict the elements of $\mathcal{V}_\omega$ to $A \subset A_\omega$.

We can apply this construction to $B := A$ and the operator-convex cone $\mathcal{V} := CP_{\mathrm{nuc}}(A, A)$ and get $\mathcal{V}_\omega|A = \mathcal{C}_{\omega\mathrm{nuc}}$.

By Lemma A.5, every ω-nuclear c.p. map $V\colon A \to A_\omega$ can be represented by a sequence of $T_1, T_2, \dots$ of nuclear c.p. maps from A into A such that $\|T_n\| \leq \|V\|$ and $T_\omega|A = V$, because $CP_{\mathrm{nuc}}(A, A)$ is operator-convex.

If $V_1, V_2, \dots$ is a sequence in $\mathcal{C}_{\omega\mathrm{nuc}}$ that converges to a map $W\colon A \to A_\omega$ in point-norm topology, then $\gamma := \sup_n \|V_n\| < \infty$, by the uniform boundedness theorem. Thus, we find nuclear c.p. maps $T_k^{(n)}$ from A to A with $\|T_k^{(n)}\| \leq \gamma$ such that $V_n = T_\omega^{(n)}|A$, (cf. Lemma A.5). By Lemma A.3 there are $S_1, S_2, \ldots \in CP_{\mathrm{nuc}}(A, A)$ with $W = S_\omega|A$. Thus $W \in \mathcal{C}_{\omega\mathrm{nuc}}$.

(2): Let κ denote the set of positive elements $b \in F(A)$ such that $a \in A \mapsto \rho(b \otimes a)$ is in $\mathcal{C}_{\omega\mathrm{nuc}}$. Then κ is a closed convex sub-cone of $F(A)_+$ by part (1). If $b \in \kappa$, $c \in F(A)$ and $d \in A^c$ with $c = d + \mathrm{Ann}(A)$ then $a \mapsto \rho(c^*bc \otimes a) = d^*\rho(b \otimes a)d$ is in $\mathcal{C}_{\omega\mathrm{nuc}}$ by (1). Thus $c^*\kappa c \subset \kappa$ for all $c \in F(A)$. It follows that κ is the positive part of the closed ideal $J_{\mathrm{nuc}} := \overline{\kappa - \kappa + i(\kappa - \kappa)}$ of $F(A)$ (by [29, prop. 1.3.8 and 1.4.5]).

(3): Let $e \in F(A)$ a positive contraction with $\|e\| = 1$. There is a positive contraction $f \in A^c$ with $e = f + \mathrm{Ann}(A)$ and $f \notin \mathrm{Ann}(A)$. Further let $a_0 \in A_+$ a strictly positive element with $\|a_0\| = 1$. Then $\rho_A(e \otimes a_0) = fa_0 \neq 0$, because $\mathrm{Ann}(A) = \mathrm{Ann}(a_0, A_\omega)$. Thus, there is a character χ on $C^*(a_0, f)$ with $\chi(a_0 f) = \|a_0 f\| \neq 0$. We extend χ to a pure state μ on $C^*(A, f)$.

By Lemma 2.5 there exist pure states $\mu_1, \mu_2, \dots$ on A and $g_1, g_2, \ldots \in A_+$ such that $\|g_n\| = 1$ and $\mu = \mu_\omega|C^*(A, f)$ and $V_g(y) := gyg = \mu(y)g^2$ for all

$y \in C^*(A, f)$ for $g := \pi_\omega(g_1, g_2, \ldots)$. In particular, $\|g\| = 1$, $g \geq 0$, $gfa_0g = \mu(fa_0)g^2 = \|fa_0\|g^2 \neq 0$. Thus $V_g|A = S_\omega|A$ for the sequence of nuclear c.p. contractions $S_1, S_2, \ldots \in CP_{\rm nuc}(A, A)$ with $S_n(a) := \mu_n(a)g_n^* g_n$.

By Proposition 1.3 (with $A = B = J$, $\mathcal{V} := CP_{\rm nuc}(A, A)$ and a, b, c, e, f, g replaced here by $0, g, 0, 0, h, 0$), there are nuclear c.p. contractions $T_1, T_2, \ldots \in CP_{\rm nuc}(A, A)$ and a positive contraction h in A^c such that $gh = g$ and $y \in A \to yh \in A_\omega$ is the restriction of T_ω to A.

Let $k := h + \mathrm{Ann}(A)$. Then $\rho_A(k \otimes y) = hy$ for $y \in A$ and $ke = (kf) + \mathrm{Ann}(A)$. It follows $k \in J_{\rm nuc}$ and $g\rho_A(ke \otimes a_0)g = ghfa_0g = ghfa_0g = gfa_0g \neq 0$, i.e., $ke \neq 0$. Hence, $J_{\rm nuc}$ is an *essential* ideal of $F(A)$.

(4): If A is nuclear, then $a \in A \to a = \rho(1 \otimes a) \in A_\omega$ is the restriction of $(\mathrm{id}_A)_\omega$ to A and id_A is nuclear. Thus $1 \in J_{\rm nuc}$, i.e. $F(A) = J_{\rm nuc}$.

Conversely, if $1 \in J_{\rm nuc}$, then there exists a sequence $(V_1, V_2, \ldots)$ of nuclear c.p. maps $V_n \colon A \to A$ such that the inclusion map $a \in A \mapsto a = \rho(1 \otimes a) \in A_\omega$ is the restriction of V_ω to A. This means that id_A can be approximated in point-norm by (convex combinations of) the nuclear c.p. maps V_n, $n = 1, 2, \ldots$. Hence, A is nuclear. □

Theorem 2.12 *Suppose that A is a separable C^*-algebra and let $F(A) := A^c/\mathrm{Ann}(A)$.*

(1) $F(A) \cong \mathbb{C}$ if and only if $A \otimes \mathcal{K} \cong \mathcal{K}$.
(2) If $F(A)$ is simple and $F(A) \ncong \mathbb{C}$, then A is simple, purely infinite and nuclear.
(3) If A is simple, purely infinite and nuclear, then $F(A)$ and A_ω are simple and purely infinite, and $A \cong A \otimes \mathcal{O}_\infty$.

Proof. Recall that $F(A)$ is unital by Corollary 1.10 , that A is simple *and* purely infinite iff, A_ω is simple by Remark 2.4, and that A is unital iff $\mathrm{Ann}(A) = \{0\}$ iff $A^c = F(A)$ by Corollary 1.10 .

(1): $F(A) = \mathbb{C}\cdot 1$ implies that A is simple (cf. Lemma 2.1). Thus $F(A) \cong \mathbb{C}$, if and only if, $A \otimes \mathcal{K} \cong \mathcal{K}$ (cf. Proposition 2.10).

(2): If $F(A)$ is simple, then A is simple by Lemma 2.1. Thus Proposition 2.10 applies to A: $F(A)$ is simple and *stably finite* if and only if $F(A) = \mathbb{C} \cdot 1$. We get that $F(A)$ is *not* stably finite if $F(A)$ is simple and $F(A) \ncong \mathbb{C}$. I.e., there is $n \in \mathbb{N}$ such that $F(A) \otimes M_n$ contains a copy of $\mathcal{O}_\infty$ unitally (because $F(A)$ is unital and *simple*).

It follows that A is purely infinite, indeed:
A is simple by Lemma 2.1, and is antiliminal by Proposition 2.10. Let $h \colon C_0((0, 1], M_n) \cong M_n \otimes C_0((0, 1]) \to A$ a *-morphism, $a := h(1_n \otimes f_0) \in A_+$ and let D the hereditary C^*-subalgebra of A generated by a. (Here $f_0(t) := t$ for $t \in [0, 1]$.) Consider the natural embedding of $\mathcal{O}_\infty \otimes^{\min} C^*(a)$ into $(F(A) \otimes M_n) \otimes^{max} C^*(a) \cong F(A) \otimes^{max} (M_n \otimes C^*(a))$ given by $\mathcal{O}_\infty \subset F(A) \otimes M_n$ and compose with $\rho \colon F(A) \otimes^{max} A \to A_\omega$. Then we get a *-monomorphism $k \colon \mathcal{O}_\infty \otimes C^*(a) \to A_\omega$ with $k(1 \otimes a) = a$. Hence, a is *properly infinite* in A_ω, i.e. for every $\varepsilon > 0$ there exist $d_1, d_2 \in A_\omega$ with

$d_i^* a d_j = \delta_{i,j}(a-\varepsilon)_+$ (cf. [24, prop. 3.3]). It implies that a is also properly infinite in A itself (use representing sequences for d_1 and d_2). Since every non-zero hereditary C^*-subalgebra of the antiliminal C^*-algebra A contains a non-zero n-homogenous element (cf. [29, lem. 6.7.1]), it follows that every non-zero element of A is properly infinite by [24, lem. 3.8]. Thus A is purely infinite by [24, lem. 4.2, prop. 5.4].

If $F(A)$ is simple then $F(A) = J_{\mathrm{nuc}}$ by Lemma 2.11(4). Hence, A is nuclear by Lemma 2.11(5).

(3): If A is simple, purely infinite and separable, then A_ω is simple and purely infinite by Remark 2.4.

For the rest of the proof it suffices to consider the case, where A is unital, because, if A is not unital, then there is a non-zero projection $p \in A$ such that $A \cong pAp \otimes \mathcal{K}$ (Zhang dichotomy, [33]). Thus $F(A) \cong F(pAp) = (pAp)^c \subset (pAp)_\omega$ by Corollary 1.10 .

If A is simple, purely infinite, separable, unital and nuclear, then, $F(A) = A^c \neq \mathbb{C}1$ by Proposition 2.10. Moreover, for $b \in A^c$ with $0 \le b \le 1$ and $\|b\| = 1$, there is an isometry $S \in A_\omega$ with $S^*bS = 1$ and $S^*aS = a$ for all $a \in A$. It follows $SS^* \in A^c$ and $S \in A^c$. Thus $F(A)$ is simple and purely infinite.

To get S, recall that the unital nuclear c.p. map $f \to f(1)$ from $C(\mathrm{Spec}(b), A) \cong C^*(b,1) \otimes A \cong C^*(b, A)$ into $A \subset A_\omega$ is approximately one-step inner (in A_ω) by [25, thm. 7.21]. Then use Proposition A.4 to pass from the approximate solutions of $x^*x - 1 = 0$, $x^*bx - 1 = 0$ and $x^*a_nx - a_n = 0$ for a dense sequence $(a_1, a_2, \ldots)$ in A_+ to the precise solution S.

It remains to show that $A \otimes \mathcal{O}_\infty \cong A$ if $F(A)$ is simple and purely infinite (and A is unital):

Then $F(A) = A^c$ contains a copy of $\mathcal{O}_\infty$ unitally. Thus $A^c \subset A_\omega$ contains a copy of $\mathcal{O}_\infty$ unitally. If the contractions $(u_1, u_2, \ldots)$ and $(v_1, v_2, \ldots)$ are representing sequences in $\ell_\infty(A)$ for s_1 and s_2 in $C^*(s_1, s_2, \ldots) = \mathcal{O}_\infty \subset A^c$ then $\lim_n \|d_n^* a d_n - a \otimes 1_2\| = 0$ for suitably chosen row-matrix $d_n := (u_{k_n}, v_{k_n}) \in M_{1,2}(A)$ and all $a \in A$. It follows $A \cong A \otimes \mathcal{O}_\infty$ by [25, prop.$\tilde{8}$.4]. □

A variation of the proof of the implication "$F(A)$ simple and not stably stably finite" $\Rightarrow$ "A purely infinite" shows also:

Remark 2.13 *Suppose that A is simple, separable and is not stably projection-less, and that $F(A)$ is not stably finite. Then A is purely infinite. (Here we do not assume that $F(A)$ is simple!)*

Proof. We can suppose that that A is unital, because A is stably isomorphic to a unital C^*-algebra B and $F(A) \cong F(B) = B^c$. On the the other hand, there is $n \in \mathbb{N}$ such that there is a *-monomorphism ψ from the Toeplitz algebra $\mathcal{T} = C^*(t; t^*t = 1)$ into $M_n(A^c) \subset M_n(A_\omega) \cong (M_n(A))_\omega$.

Since $\mathcal{T}$ is (weakly) semi-projective, there is a also *-monomorphism $\varphi\colon \mathcal{T} \to M_n(A)$. In particular, $\mathcal{K} \otimes A$ contains an infinite projection q, and A is antiliminal.

Let $0 \neq a \in A_+$. Since A is antiliminal, there is a non-zero *-morphism $h\colon C_0(0,1] \otimes M_n \to \overline{aAa}$ by the Glimm halving lemma [29, lem. 6.7.1]. Let $d := h(f_0 \otimes e_{1,1}) \in A_+$ and $D := \overline{dAd}$, and recall that $\mathcal{K} \subset \psi(\mathcal{T}) \subset M_n(F(A))$. Thus,

$$\mathcal{K} \otimes A \cong (id_n \otimes \rho)(\mathcal{K} \otimes D) \subset M_n \otimes D_\omega \cong M_n(D)_\omega \,.$$

Since $\mathcal{K} \otimes A$ contains an infinite projection q, $M_n(D)_\omega$ contains an infinite projection p. Since the defining relations for infinite projections are semi-projective, we get that $M_n(D) \cong \overline{h(f_0 \otimes 1_n)Ah(f_0 \otimes 1_n)}$ and $\overline{aAa}$ contain infinite projections. □

Remark 2.13 suggests the question:

Question 2.14 *Suppose that A is simple, separable and stably projection-less. Is 1 finite in $F(A)$?*

Remark 2.15 *Let $\mathcal{A}$ denote the simple purely infinite reduced free product C^*-algebra of two matrix-algebras with respect to non-central states as considered in [14]. Then $F(\mathcal{A})$ is finite and is not* not *simple.*

Proof. $\mathcal{A}$ is simple, purely infinite, unital and exact, but $F(A) = \mathcal{A}^c$ does not contain a non-unitary isometry (because $\mathcal{A}^c$ does not contain non-trivial projections by [14]). $F(\mathcal{A})$ is not simple by Theorem 2.12, because $\mathcal{A}$ is not nuclear. □

Proposition 2.10 implies that $C^*_{\mathrm{red}}(F_2)^c = F(C^*_{\mathrm{red}}(F_2))$ is a *non-separable* algebra, moreover, its maximal Abelian C^*-subalgebras have perfect maximal ideal spaces and are not separable. $F(C^*_{\mathrm{red}}(F_2))$ is stably finite by Remark 2.13. The natural *-morphism from $C^*_{\mathrm{red}}(F_2)^c$ to the commutant $\cong \mathbb{C}$ of $C^*_{\mathrm{red}}(F_2)$ in the von-Neumann ultrapower $VN(F_2)^\omega$ defines a character on $C^*_{\mathrm{red}}(F_2)^c$. Thus $C^*_{\mathrm{red}}(F_2)^c$ is not simple (that also follows from Theorem 2.12).

Question 2.16 *Is $C^*_{\mathrm{red}}(F_2)^c$ non-Abelian? Is its essential ideal J_{nuc} simple?*

Remark 2.17 *Every separable nuclear C^*-algebra is in the UCT–class, if and only if,* $[1] = 0$ *in* $K_0(F(D))$ *for every simple p.i.s.u.n. algebra D with* $K_*(D) = 0$.

Proof. For simple p.i.s.u.n. algebras D holds that $D \cong \mathcal{O}_2$ if $[1] = 0$ in $K_0(F(D))$, i.e. if $\mathcal{O}_2$ is unitally contained in the simple purely infinite algebra $F(D) = D^c$ (cf. [23], or end of [20], or [21], or Section 4 for different proofs).

On the other hand: For every separable C^*-algebra A there are a separable commutative C^*-algebra C and a semisplit extension

$$0 \to SA \otimes \mathcal{K} \to \mathcal{E} \to C \otimes \mathcal{K} \to 0$$

with $K_*(\mathcal{E}) = 0$ (cf. [3, sec. 23]). $\mathcal{E}$ is in the UCT-class iff A is in the UCT-class. $\mathcal{E}$ is nuclear (respectively exact) iff A is nuclear (respectively exact).

For every nuclear (respectively exact) separable C^*-algebra $\mathcal{E}$ there is a KK-equivalent nuclear (respectively exact) separable unital C^*-algebra B, which contains a copy of $\mathcal{O}_2$ unitally and is KK-equivalent to $\mathcal{E}$ (the reader can find a suitable C^*-subalgebra of a corner of $(\mathcal{E}+\mathbb{C}\cdot 1)\otimes\mathcal{O}_\infty$). Let $h_0\colon\ B \hookrightarrow \mathcal{O}_2 \subset B$ an unital embedding of B into $\mathcal{O}_2$, and let $h := \mathrm{id}_B \oplus h_0 \in \mathrm{End}(B)$ (Cuntz addition). Then $h\colon\ B \to B$ satisfies $[h]_{KK} = [id_B]_{KK}$, and it is easy to see that the inductive limit

$$D := \mathrm{indlim}_n(h_n\colon\ B \to B)$$

with $h_n := h$ is simple, p.i. and nuclear (respectively exact). Since the unitary group of $\mathcal{O}_2$ is a contractible space, one can construct explicitly a unital *-morphism $k\colon\ D \to C_b([1,\infty),B)/C_0([1,\infty),B)$ that has a u.c.p. lift $V\colon\ D \to C_b([1,\infty),B)$ and that is an “inverse" of the unital embedding $h\colon\ B \to B \subset D \subset C_b([1,\infty),D)/C_0([1,\infty),D)$ with respect to an “unsuspended" and cp-liftable variant of E-theory. (This is the *crucial point* of the proof, because one has to overcome the discontinuity of the KK-functor with respect to inductive limits, cf. [21, chp. 11] for more details.)

It follows, that $B \to D$ define a KK-equivalences. Thus, a separable nuclear C^*-algebra A is in the UCT-class, if and only if, the above constructed (simple) p.i.s.u.n. algebra D with $K_*(D) = K_*(B) = K_*(\mathcal{E}) = 0$ is isomorphic to $\mathcal{O}_2$, and this is the case, if and only if, $[1] = 0$ in $K_0(F(D))$. □

Similar arguments show:

Remark 2.18 *$K_0(D \otimes D) = 0$ for all (simple) p.i.s.u.n. algebras D with $K_*(D) = 0$, if and only if, the Künneth theorem on tensor products (KTP) for the calculation of $K_*(B_1 \otimes B_2)$ holds for every pair (B_1, B_2) of* nuclear *C^*-algebras.*

There are separable purely infinite unital non-separable C^*-algebras A with $A^c \cong \mathbb{C}$ (e.g. the Calkin algebra by Corollary 2.21). This comes from the following Lemma and from Voiculescu's description of the neutral element of $\mathrm{Ext}(B)$ for separable B (cf. proof of Proposition 2.20).

Lemma 2.19 *Let B a separable unital C^*-algebra. There exist a unital C^*-algebra D, a unital *-monomorphism $\eta\colon\ B \to D$ and a projection $p \in D$ such that*

$$\|(1-p)\eta(b)p\| = \|p\eta(b) - \eta(b)p\| = \mathrm{dist}(b, \mathbb{C}\cdot 1)$$

for every $b \in B$.

Proof. Let $D := B * E$ the unital full free C^*-algebra product of B and of $E := C^*(1, p = p^2 = p^*) \cong \mathbb{C} \oplus \mathbb{C}$. Then $\eta\colon\ b \mapsto b * 1$ and $\theta\colon\ e \to 1 * e$ are unital *-monomorphisms from B (respectively from E) into D. We identify $e \in E$ with $\theta(e)$. Note that, for all $b \in B$,

$$\max(\|(1-p)\eta(b)p\|,\ \|p\eta(b)(1-p)\|) = \|p\eta(b) - \eta(b)p\| \le \mathrm{dist}(b, \mathbb{C}\cdot 1).$$

Let $b \in B \setminus \mathbb{C} \cdot 1$, i.e. $\mathrm{dist}(b, \mathbb{C} \cdot 1) > 0$. Since $|z| \leq \|b - z1\| + \|b\|$, there exists $z_0 \in \mathbb{C}$ with $|z_0| \leq 2\|b\|$ such that $\|b - z_0 1\| = \mathrm{dist}(b, \mathbb{C} \cdot 1)$. $\mathrm{dist}(b, \mathbb{C} \cdot 1)$ is the norm of $b + \mathbb{C} \cdot 1$ in $B/(\mathbb{C} \cdot 1)$. Thus, there exists a linear functional φ on B with $\varphi(1) = 0$, $\|\varphi\| = 1$ and $\varphi(b - z_0 1) = \|b - z_0 1\|$. If we use the polar-decomposition $\varphi = |\varphi|(u\cdot)$ of φ in $B^* = (B^{**})_*$, cf. [29, prop. 3.6.7], we can see that there are a unital *-representation $\lambda\colon B \to \mathcal{L}(\mathcal{H})$ and vectors $x, y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$ such that $\varphi(c) = \langle \lambda(c)x, y\rangle$ for all $c \in B$. It follows $x \perp y$ and $\lambda(b - z_0 1)x = \|b - z_0 1\| y$. Let $q \in \mathcal{L}(\mathcal{H})$ denote the orthogonal projection onto $\mathbb{C}x$. Then $(1 - q)\lambda(b)qx = \|b - z_0 1\| y$. Thus

$$\mathrm{dist}(b, \mathbb{C} \cdot 1) \leq \|(1 - q)\lambda(b)q\| \leq \|(1 - p)\eta(b)p\|$$

because there is a unital *-morphism $\kappa\colon D \to \mathcal{L}(\mathcal{H})$ with $\kappa(p) = q$ and $\kappa(\eta(b)) = \lambda(b)$. □

Proposition 2.20 *For every separable unital C^*-subalgebra B of the Calkin algebra $Q := \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ (on $\mathcal{H} \cong \ell_2(\mathbb{N})$) there is a projection $P \in Q$ with $\|Pb - bP\| = \mathrm{dist}(b, \mathbb{C} \cdot 1)$ for all $b \in B$.*

Proof. Let D, $\eta\colon B \to D$ and $p \in D$ as Lemma 2.19. D can be unitally and faithfully represented on $\mathcal{H} := \ell_2(\mathbb{N})$ such that $D \cap \mathcal{K} = \{0\}$. Let $s_1, s_2 \in \mathcal{L}(\mathcal{H})$ two isometries with $s_1 s_1 * + s_2 s_2^* = 1$, $\pi\colon t \in \mathcal{L}(\mathcal{H}) \mapsto t + \mathcal{K} \in Q$ denotes the quotient map. There is a unitary $U \in Q$ with $U^* b U = \pi(s_1) b \pi(s_1)^* + \pi(s_2 \eta(b) s_2^*)$ for $b \in B$, by the generalized Weyl–von-Neumann theorem of Voiculescu, cf. [2]. Thus $P := U\pi(s_2 p s_2^*)U^*$ is a projection in Q that satisfies $\|Pb - bP\| = \mathrm{dist}(b, \mathbb{C} \cdot 1)$ for all $b \in B$. □

Proposition 2.20 implies:

Corollary 2.21 *Let $Q := \mathcal{L}(\ell_2)/\mathcal{K}(\ell_2)$. Then $Q^c = \mathbb{C} \cdot 1$.*

Proof. Let $b = \pi_\omega(b_1, b_2, \ldots) \in Q_\omega$ for $(b_1, b_2, \ldots) \in \ell_\infty(Q)$, B the unital C^*-subalgebra generated by $b_1, b_2, \ldots$ and $P \in Q$ as in Proposition 2.20. Then

$$Pb - bP = \pi_\omega(Pb_1 - b_1 P, Pb_2 - b_2 P, \ldots)$$

and $\|Pb - bP\| = \lim_\omega \mathrm{dist}(b_n, \mathbb{C} \cdot 1)$. It follows $b \in \mathbb{C} \cdot 1 \cong (\mathbb{C} \cdot 1)_\omega$ if $Pb = bP$. □

Question 2.22 *Is $\mathcal{L}(\ell_2)^c = \mathbb{C} \cdot 1$?*

The question leads to a study of the positive elements in $\mathrm{Ann}(\mathcal{K}, \mathcal{K}_\omega)$: Note that $\mathcal{L}(\ell_2)^c \subset (\mathcal{K}(\ell_2) + \mathbb{C} \cdot 1)_\omega$ by Cor. 2.21, and that $F(\mathcal{K} + \mathbb{C} \cdot 1) = (\mathcal{K} + \mathbb{C} \cdot 1)^c = \mathrm{Ann}(\mathcal{K}, \mathcal{K}_\omega) + \mathbb{C} \cdot 1$, because $F(\mathcal{K}) \cong F(\mathbb{C}) \cong \mathbb{C}$.

Remark 2.23 *If A is a simple C^*-algebra, then for every $g, h \in (A_\omega)_+$ with $\|g\| = \|h\| = 1$ there is $z \in A_\omega$ with $\|z\| = 1$ and $zz^* g = zz^*$, $z^* z h = z^* z$. In particular,* $\mathrm{Ann}(A)$ *does not contain a non-zero closed ideal* J of A_ω *if A is simple.*

3 The Invariant $\mathrm{cov}(F(A))$ and Applications

Here we consider the case where A is separable and $F(A)$ contains a full simple C^*-algebra B of dimension $\mathrm{Dim}(B) > 1$. We show below that (in this case) A is strongly purely infinite if A is weakly purely infinite. Other considerations of this section are concerned with a (sufficient) condition on $F(A)$ under which A is weakly purely infinite if every (extended) lower semi-continuous 2-quasi-trace on A_+ is trivial (i.e. takes only the values 0 and ∞, cf. Proposition 3.7). The main result of this section is Theorem 3.10.

We say that $X \subset B_+$ is *full* if the ideal of B generated by X is dense in B, $b \in B_+$ is full if $X := \{b\}$ is full, and a *-morphism $h\colon C \to B$ is full if $h(C_+)$ is full in B.

Recall that a positive contraction $b \in B_+$ is k*-homogenous* if there is a *-morphism $h\colon C_0((0,1]) \otimes M_k \to B$ such that $h(f_0 \otimes 1_k) = b$. (Here $f_0(t) := t$ for $t \in (0,1]$, and 0 is k-homogenous for every $k \in \mathbb{N}$ by definition.)

Definition 3.1 *We define* $\mathrm{cov}(B,m) \in \mathbb{N} \cup \{+\infty\}$ *for a unital* C^**-algebra* B *(and* $m > 1$*) as the minimum of the set of* $n \in \mathbb{N}$ *with the property that there are* $a_1, \ldots, a_n \in B_+$ *and* $d_1, \ldots, d_n \in B$ *with* $\sum_j d_j^* a_j d_j = 1$ *and that* a_j *is the sum* $a_j = \sum_{i=1}^{l_j} a_{j,i}$ *of mutually orthogonal* $k_{j,i}$*-homogenous elements* $a_{j,i} \in B_+$ *with* $k_{j,i} \geq m$ *for* $j = 1, \ldots, n$ *and* $i = 1, \ldots, l_j$*. (The minimum of an empty subset of* $\mathbb{N}$ *is considered as* $+\infty$*.) In other words:*
$\mathrm{cov}(B,m) \leq n < \infty$*, if and only if, there are finite-dimensional* C^**-algebras* $F_1, \ldots, F_n$*, *-morphisms* $h_j\colon C_0((0,1]) \otimes F_j \to B$ *and* $d_1, \ldots, d_j$ *such that every irreducible representation of* F_j *is of dimension* $\geq m$ *and* $1 = \sum_j d_j^* h_j(f_0 \otimes 1) d_j$ *for* $j = 1, \ldots, n$*.*

We define:

$$\mathrm{cov}(B) := \sup_m \mathrm{cov}(B,m).$$

One can replace the F_j in the definition of $\mathrm{cov}(B,m)$ by those unital C^*-subalgebras $G_j \subset F_j$ which have, moreover, only irreducible representations $D\colon G_j \to \mathcal{L}(\mathcal{H})$ of dimension $m \leq \mathrm{Dim}(\mathcal{H}) < 2m$ and a center $\mathcal{Z}(G_j)$ of dimension $< m$.

It is useful to note that the definition of $\mathrm{cov}(B,m)$ can be described by weakly semi-projective relations (e.g. for the study of cov of ultrapowers or of continuity properties of $B \mapsto \mathrm{cov}(B)$, Remark 3.3 and below):

Remark 3.2 *We can suppose that the* $d_1, \ldots, d_n$ *and* $h_j\colon C_0((0,1]F_j) \to B$ *of Definition 3.1 satisfy in addition the weakly semi-projective relations*

$$d_1^* d_1 + \ldots + d_n^* d_n = 1 \qquad \text{and} \qquad h_j(f_0 \otimes 1) d_j = d_j$$

for $j = 1, \ldots, n := \mathrm{cov}(B,m)$*.*

It follows:
$\mathrm{cov}(B) \leq n$, if and only if, there is a unital *-morphism from the "locally" weakly semi-projective C^*-algebra $\mathcal{A}_n := \mathcal{A}_{n,1} * \mathcal{A}_{n,2} * \cdots$ into B.

Here $\mathcal{A}_{n,k}$ denotes the (weakly semi-projective) universal unital C^*-algebra generated by n copies $h_j(C_k) \subset B$ of $C_k := C_0((0,1],(M_2 \oplus M_3)^{\otimes k})$ and elements $d_1,\ldots,d_n$ with relations $d_1^*d_1+\ldots+d_n^*d_n = 1$ and $h_j(f_0\otimes 1)d_j = d_j$ for $j = 1,\ldots,n$, and $\mathcal{A}_{n,1}*\mathcal{A}_{n,2}*\mathcal{A}_{n,3}*\cdots$ means the unital universal (="full") free product of unital C^*-algebras.

Proof. To get weakly semi-projective relations, let $k_j\colon C_0((0,1])\otimes F_j \to B$ and $e_1,\ldots,e_n$ such that $1 = \sum_j e_j^* k_j(f_0\otimes 1)e_j$ (where F_j is finite-dimensional and every irreducible representation of F_j is of dimension $\geq m$ for $j = 1,\ldots,n$). Then $1/2 < g := \sum_j e_j^* k_j((f_0-\delta)_+\otimes 1)e_j \leq 1$ for suitable $\delta \in (0,1)$. Let $d_j := k_j((f_0-\delta)_+\otimes 1)^{1/2}e_j g^{-1/2}$ then $d_1^*d_1+\ldots+d_n^*d_n = 1$. There is a *-morphism $\psi\colon C_0(0,1] \to C_0(0,1]$ with $\psi(f_0) = g_\delta$ where $g_\delta(t) := \min(t/\delta,1)$. Let $h_j := k_j \circ (\psi\otimes \mathrm{id}_{F_j})$, then $h_j(f_0\otimes 1)d_j = d_j$.

The new relations are away from the old relations, but they do the same job (for the definition of $\mathrm{cov}(B,n)$) and they are *weakly semi-projective* in the category of unital C^*-algebras:
The relation $\sum d_j^*d_j = 1$ is semi-projective in the category of unital C^*-algebras and the defining relations of $C_0((0,1],F_j)$ are even projective in the category of all C^*-algebras (cf. [27, thm. 10.2.1], [28], or the elementary proof in [6, sec. 2.3]). Let $d_1,\ldots,d_n$ contractions with $\sum_j d_j^*d_j = 1$, and $h_j\colon C_0((0,1],F_j) \to B$ *-morphisms with $\|h_j(f_0\otimes 1_{F_j})d_j - d_j\| < \delta^2/n$ for some $\delta \in (0,1/2)$. Then

$$\|1-\delta^{-1}\sum d_j^*c_jd_j\| < \delta < 1/2$$

for $c_j := h_j(f_0\otimes 1)$, because $\|d_j\| \leq 1$ and $\delta - \max(0,t-(1-\delta)) \leq 1-t$ for $0<\delta<1$, $t\in[0,1]$ (i.e. because $1-\delta^{-1}(c_j-(1-\delta))_+ \leq \delta^{-1}(1-c_j)$).

Let $g_0 := \delta^{-1}(f_0-(f_0-(1-\delta)))_+$, $w := (\sum_j \delta^{-1}d_j^*c_jd_j)^{-1/2}$, $d_j' := \delta^{-1/2}c_j^{1/2}d_jw$ and define *-morphisms $h_j'\colon C_0((0,1],F_j) \to B$ by $h_j'(f_0^k\otimes x) := h_j(g_0^k\otimes x)$ for $k\in\mathbb{N}$, $x\in F_j$ and $j = 1,\ldots,n$.

The new system $d_j' \in B$, $h_j'\colon C_0((0,1],F_j) \to B$ satisfies $h_j'(f_0\otimes 1)d_j' = d_j'$ and the canonical generators differ form d_j and the old images by h_j of the canonical generators of $C_0((0,1],F_j)$ by $\|f_0-g_0\| < \delta^{1/2}$ and $\|w-1\| < \delta^{1/2}$.

The use of $(M_2\oplus M_3)^{\otimes k}$ instead of F_j (with minimal dimension of irreducible representations $\geq m$ in an asymptotic sense) is possible, because every irreducible representation of $(M_2\oplus M_3)^{\otimes k}$ has dimension $\geq 2^k$ and there is a unital *-morphism from $(M_2\oplus M_3)^{\otimes k}$ into M_ℓ for *all* $\ell > 6^k$, because $1 = 2^kx - 3^ky = 3^k(2^k-y) - 2^k(3^k-x)$ with $1<x<3^k$ and $1<y<2^k$ (but 6^k+1 is not the smallest value for ℓ with the property that all numbers $\ell,\ell+1,\ell+2\ldots,$ are sums $\sum_{0\leq j\leq k} n_j2^j3^{k-j}$ with $n_j\in\mathbb{N}\cup\{0\}$). □

One can read off some properties of $\mathrm{cov}(B,m)$ and $\mathrm{cov}(B)$ straight from Definition 3.1 and Remark 3.2:

Remark 3.3 *The maps $(B,m) \mapsto \mathrm{cov}(B,m)$ and $B \mapsto \mathrm{cov}(B)$ on unital C^*-algebras B have the properties:*

(1) $\mathrm{cov}(B,m) \le \mathrm{cov}(B,m+1)$,
(2) $\mathrm{cov}(C,m) \le \mathrm{cov}(B,m)$ *if there exist a *-morphism* $\psi\colon B \to C$ *such that* $\psi(1) = 1$, *or that* $\psi(1_B)$ *is properly infinite and is full in* C.
In particular, $\mathrm{cov}(\mathcal{O}_\infty, m) = \mathrm{cov}(\mathcal{O}_2, m) = \mathrm{cov}(M_{2^\infty}, m)\mathrm{cov}(M_{2^m}, m) = 1$ *for* $m > 1$.
(3) *If* $B_1, B_2, \ldots$ *is a sequence of unital* C^**-algebras, then, for every* $m \in \mathbb{N}$, $\mathrm{cov}(\prod_\omega \{B_1, B_2, \ldots), m\} = \lim_\omega \mathrm{cov}(B_n, m)$ *and* $\mathrm{cov}(\prod_\omega \{B_1, B_2, \ldots\}) = \lim_\omega \mathrm{cov}(B_n)$. [1]
In particular, $\mathrm{cov}(B_\omega, m) = \mathrm{cov}(B, m)$, *and* $\mathrm{cov}(B_\omega) = \mathrm{cov}(B)$.
(4) $\mathrm{cov}(B,m) = \inf_n \mathrm{cov}(B_n, m)$, $\mathrm{cov}(B) = \sup_m \inf_n \mathrm{cov}(B_n, m) \le \sup_n \mathrm{cov}(B_n)$, *if* $B_1 \subset B_2 \subset \cdots \subset B$ *are unital* C^**-subalgebras with* $\bigcup_n B_n$ *dense in* B.
(5) *Suppose that* 1_B *is finite. Then* $\mathrm{cov}(B,m) = 1$, *if and only if, there are a* C^**-algebra* A_m *of finite dimension and a unital *-morphism* $h_m\colon A_m \to B$, *where every irreducible representation of* A_m *has dimension* $\ge m$.
(6) $\mathrm{cov}(B) = 1$ *if* 1_B *is properly infinite.*
(7) $\mathrm{cov}(B) = \mathrm{cov}(B,m) = \infty$ *if every irreducible representation of* B *has dimension* $\le m-1$.
(8) $\mathrm{cov}(B,m) < \infty$ *for every* $m \in \mathbb{N}$ *if* B *is strictly antiliminal.*
(9) *If* B *has real rank zero, then* $\mathrm{cov}(B,m) = 1$ *if and only if there exist* $1 \le p < m$, $F = M_{k_1} \oplus \ldots \oplus M_{k_p} \subset B$ *with* $m \le k_j < 2m$ *for* $j = 1, \ldots, p$ *and an isometry* $d \in B$ *with* $1_F d = d$.
(10) *Every separable* C^**-subalgebra* $B_1 \subset B$ *of a unital* C^**-algebra* B *is contained in a unital* C^**-subalgebra* $1_B \in B_2 \subset B$ *with* $\mathrm{cov}(B_2, m) = \mathrm{cov}(B,m)$ *for all* $m \in \mathbb{N}$.

Proof. (1), (2), and (7) follow immediately from Definition 3.1. (5) and (9) follow from Remark 3.2. (3): Use Remark 3.2 and (2). (4): Use (2) for $\le$ and (i,iii) for $\ge$. (6): Use (2). (8): Use the Glimm halving lemma [29, lem. 6.7.1] to see that 1_B is majorized by a finite sum of m-homogenous positive contractions. (10): Use Remark 3.2 and (4). □

Proposition 3.4 *Suppose that* A *is an inductive limit* $\mathrm{indlim}(h_n\colon A_n \to A_{n+1})$ *of separable* C^**-algebras* $A_1, A_2, \ldots$. *Then*

$$\mathrm{cov}(F(A), m) \le \liminf_{n\to\infty} \mathrm{cov}(F(A_n), m)\,.$$

In particular, $\mathrm{cov}(F(A)) \le \liminf \mathrm{cov}(F(A_n))$.

Proof. Remark 3.3(4) is not applicable, because the $F(A_n)$ are not related to $F(A)$. But Proposition 1.14 works:
$\mathrm{cov}(F(A_n/I)) \le \mathrm{cov}(F(A_n))$ for closed ideals I of A_n by Remark 3.3(2), because $F(A_n/I)$ is a quotient of $F(A_n)$ by Remark 1.15. Thus, we may suppose that $A_1, A_2, \ldots \subset A$ and $\bigcup_n A_n$ is dense in A.

[1] We extended $\lim_\omega$ to all sequences $(\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)$ with $\alpha_n \in [0, \infty]$.

By Proposition 1.14, for every $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$ and for every separable unital C^*-subalgebra $E \subset \prod_\omega \{F(A_1), F(A_2), \ldots\}$, there is a unital *-morphism $E \to F(A)$. Thus $\mathrm{cov}(F(A)) \leq \mathrm{cov}(E)$ by Remark 3.3(2). E can be found such that $\mathrm{cov}(E, m) = \mathrm{cov}(\prod_\omega \{F(A_1), F(A_2), \ldots\}, m)$ for every $m \in \mathbb{N}$ by Remark 3.3(10). Now apply Remark 3.3(3) and note that for $\alpha_1, \alpha_2, \ldots \in [0, \infty]$ there is a free ultrafilter $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$ with $\lim_\omega \alpha_n = \liminf_{n \to \infty} \alpha_n$. □

Proposition 3.5 *If a unital nuclear separable C^*-algebra B has decomposition rank $\mathrm{dr}(B) < \infty$ (cf. [26, def. 3.1]) and if B has no irreducible representation of finite dimension, then $\mathrm{cov}(B) \leq \mathrm{dr}(B) + 1$.*

Proof. This follows easily from the definition of the decomposition rank [26, def. 3.1] by [26, prop. 5.1], which implies that the c.p. contractions $\varphi_{r_i} \colon M_{r_i} \to B$ of strict order zero arising in n-decomposable c.p. approximations $\varphi \colon \bigoplus_{i=1}^s M_{r_i} \to B$ and $\psi \colon B \to \bigoplus_{i=1}^s M_{r_i}$ of [26, def. 3.1] can be chosen such that (eventually) $\min\{r_1, \ldots, r_s\} \geq q$ if $\psi \circ \varphi \to \mathrm{id}_B$ (in point-norm) and B has no irreducible representation of dimension $\leq q$.

Indeed, suppose that $\varphi_n \colon C_n \oplus D_n \to B$ and $\psi_n \colon B \to C_n \oplus D_n$ are completely positive contractions with suitable C^*-algebras C_n and D_n such that $\varphi_n \circ \psi_n$ tends to id_B in point-norm, $\lim_n \|\psi_n(b^*b) - \psi_n(b^*)\psi_n(b)\| = 0$ for all $b \in B$, ψ_n is unital and every irreducible representation of C_n has dimension $\leq q$. Then the ultrapower $C := \prod_\omega \{C_1, C_2, \ldots\}$ has only irreducible representations of dimension $\leq q$ and the restriction to B of the ultrapower $U \colon B_\omega \to C$ of the completely positive contractions $p_1 \circ \psi_n \colon B \to C_n$ is a unital *-morphism from B into C. The latter contradicts that B has no irreducible representation of dimension $\leq q$. □

Recall that a quasi-trace $\tau \colon A_+ \to [0, \infty]$ is *trivial* if it takes only the values 0 and $+\infty$. The following is a reformulation of [24, prop. 5.7].

Remark 3.6 *Suppose that every lower semi-continuous 2-quasi-trace on A_+ is trivial. Then, for every $n \in \mathbb{N}$, $a \in A_+ \setminus \{0\}$ and $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ there are $d_1, \ldots, d_n$ in $M_k \otimes A$ such that $d_i^*(1_k \otimes a)d_j = \delta_{i,j}(1_k \otimes (a - \varepsilon)_+)$ for $i, j = 1, \ldots, n$.*

Proposition 3.7 *If $\mathrm{cov}(F(A)) < \infty$ and if every lower semi-continuous 2-quasi-trace on A_+ is trivial, then A is weakly purely infinite.*

Proof. Let $m := \mathrm{cov}(F(A))$ and $n := 2m$. Below we show that, for $a \in A_+$ and $\varepsilon > 0$, there exists a matrix $V = [v_{j,q}]_{m,n} \in M_{m,n}(A_\omega)$ such that $V^*(a \otimes 1_m)V = (a - \varepsilon)_+ \otimes 1_n$. It follows that A is pi-m in the sense of [25, def. 4.3] (because one can use representing sequences and the isomorphism $M_{m,n}(A_\omega) \cong (M_{m,n}(A))_\omega$). Thus A is weakly purely infinite.

Let $k_0 \in \mathbb{N}$ as in Remark 3.6 for $a \in A_+$ and $\varepsilon > 0$. We find finite-dimensional C^*-algebras $F_1, \ldots, F_m$, *-morphisms $h_j \colon C_0((0,1]) \otimes F_j) \to F(A)$ and elements $g_j \in F(A)$ such that $\sum_j g_j^* b_j g_j = 1$ for $b_j := h_j(f_0 \otimes 1_{F_j})$, and that F_j has only irreducible representations of dimension $\geq k_0$ for $j = 1, \ldots, m$. (We allow $b_j = 0$ for $\mathrm{cov}(F(A), k_0) \leq j \leq m$, to simplify notation.)

For every $j = 1, \ldots, m$ we find by Remark 3.6 $d_{j,1}, \ldots, d_{j,n} \in F_j \otimes A$ such that, for $1 \leq j \leq m$ and $1 \leq p, q \leq n$

$$d_{j,p}^*(1_{F_j} \otimes a) d_{j,q} = \delta_{p,q}(1_{F_j} \otimes (a - \varepsilon)_+) .$$

Since $g_j \otimes 1 \in \mathcal{M}(F(A) \otimes A)$, we can define, for $j = 1, \ldots, m$ and $q = 1, \ldots, n = 2m$,

$$v_{j,q} := \rho(h_j \otimes \mathrm{id}_A(f_0 \otimes d_{j,q})(g_j \otimes 1)) .$$

A straight calculation shows

$$v_{j,p}^* a v_{j,q} = \delta_{p,q} \rho \left(g_j^* b_j g_j \otimes (a - \varepsilon)_+ \right) ,$$

i.e. $V := [v_{j,q}]_{m,n}$ is as desired. □

Now we study situations where we can deduce strong pure infiniteness from weak pure infiniteness.

Lemma 3.8 *If A is purely infinite and $F(A)$ contains two orthogonal full hereditary C^*-subalgebras, then A is strongly purely infinite.*

Proof. Let $a, b \in A_+$ and $\varepsilon > 0$, $\delta := \varepsilon/2$. If $E_1, E_2 \subset F(A)$ are orthogonal full hereditary C^*-subalgebras, there are $e_i \in (E_i)_+$ and $g_j, h_k \in F(A)$ ($i = 1, 2$, $j = 1, \ldots, m$, $k = 1, \ldots, n$) such that $1 = \sum_j g_j^*(e_1)^2 g_j$ and $1 = \sum_k h_k^*(e_2)^2 h_k$. Thus, $a^2 = \rho(1 \otimes a^2)$ (respectively b^2) is in the ideal of A_ω generated by $\rho(e_1 \otimes a)$ (respectively $\rho(e_2 \otimes b)$), because, e.g. $1 \otimes a^2$ is in the ideal of $F(A) \otimes^{\max} A$ generated by $e_1 \otimes a$. Let $u_i \in (A^c)_+ \subset A_\omega$ with $e_i = u_i + \mathrm{Ann}(A)$. Then $u_1 a b u_2 = \rho(e_1 e_2 \otimes ab) = 0$ and a^2 (respectively b^2) is in the closed ideal of A_ω generated by $u_1 a^2 u_1 = \rho((e_1)^2 \otimes a^2)$ (respectively $u_2 b^2 u_2$).

Since A is purely infinite, A_ω is again purely infinite, cf. [24]. It follows that there are $f_1, f_2 \in A_\omega$ such that $f_1 u_1 a^2 u_1 f_1 = (a^2 - \delta)_+$ and $f_2 u_2 b^2 u_2 f_2 = (b^2 - \delta)_+$.

With $v_i := f_i u_i$ holds $\|v_1^* a^2 v_1 - a^2\| < \varepsilon$, $\|v_2^* b^2 v_2 - b^2\| < \varepsilon$ and $v_1^* a b v_2 = 0$ in A_ω. With help of representing sequences for v_1 and v_2 in $\ell_\infty(A)$ we find $d_1, d_2 \in A$ with $\|d_1^* a^2 d_1 - a^2\| < \varepsilon$, $\|d_2^* b^2 d_2 - b^2\| < \varepsilon$ and $\|d_1^* a b d_2\| < \varepsilon$. This means that A is strongly purely infinite, cf. [6], [25]. □

Lemma 3.9 *If $F(A)$ contains a full 2-homogenous element, then A has the global Glimm halving property of [5] (cf. also [6]).*

If, in addition, A is weakly purely infinite, then A is strongly purely infinite.

Proof. Let $a \in A_+$, $\varepsilon \in (0, 1)$, $\delta := \varepsilon^2/2$ and $D := \overline{aAa}$. By assumption, there exists $b \in F(A)$ and $d_1, \ldots, d_n \in F(A)$ with $b^2 = 0$ and $\sum_j d_j^* b^* b d_j = 1$.

Let $e_j := \rho(d_j \otimes a^{1/2})$, $c \in A^c$ with $b = c + \mathrm{Ann}(A)$ and $f := ca = \rho(b \otimes a^{1/2})$. Then $f^2 = 0$ and $a^2 = \sum_j e_j f^* f e_j$. f and $e_1, \ldots, e_n$ are in the hereditary C^*-subalgebra of A_ω generated by a, in particular they are in D_ω. Let $h = (h_1, h_2, \ldots) \in \ell_\infty(D)$ self-adjoint with $\pi_\omega(h) = f^*f - ff^*$, $g = (g_1, g_2, \ldots) \in \ell_\infty(D)$ with $\pi_\omega(g) = f$, and let $u_k := (h_k)_-^{1/k} g_k (h_k)_+^{1/k}$ for $k = 1, 2, \ldots$. Then $u_k \in D$, $u_k^2 = 0$ and $\pi_\omega(u_1, u_2, \ldots) = f$.

There exists $k \in \mathbb{N}$ and $v_1, \ldots, v_n \in D$ such that $\|a^2 - \sum_j v_j^* u_k^* u_k v_j\| < \delta$ (use representing sequences for $e_1, \ldots, e_n \in D_\omega$).

By [25, lem. 2.2] there is a contraction $z \in A$ such that $\sum_j w_j^* u_k^* u_k w_j = (a-\varepsilon)_+$ for $w_j := v_j z h(a)$ with $h(t) := \max(0, t-\varepsilon)^{1/2} / \max(0, t^2 - \delta)^{1/2}$ on $[0, \infty]$. Hence $(a-\varepsilon)_+$ is in the ideal generated by u_k.

Thus A has the global Glimm halving property of [5].

By [6] (and [5]) A is purely infinite, if and only if, A is weakly purely infinite and has the global Glimm halving property. Then A is moreover strongly purely infinite, by Lemma 3.8. □

Theorem 3.10 *If every lower semi-continuous* 2*-quasi-trace on* A_+ *is trivial and if* $F(A)$ *contains a simple* C^**-subalgebra* B *with* $1 \in B$ *and*

$$\operatorname{cov}(B \otimes^{\max} B \otimes^{\max} \cdots) < \infty,$$

then A *is strongly purely infinite.*

Proof. There is a unital *-morphism from $B \otimes^{\max} B \otimes^{\max} \cdots$, into $F(A)$ by Corollary 1.13. Since $\operatorname{cov}(B \otimes^{\max} B \otimes^{\max} \cdots) < \infty$, it follows $B \neq \mathbb{C}$ and $\operatorname{cov}(F(A)) < \infty$, cf. Remarks 3.3(ii,vii).

Thus Proposition 3.7 applies, and A is weakly purely infinite. The Glimm halving lemma (cf. [29, lem. 6.7.1]) applies to B or to $B \otimes B \otimes \cdots$ if $B \cong M_n$ with $n > 2$. Thus Lemma 3.9 applies, and A is strongly purely infinite. □

Let $\mathcal{I}(m,n) \subset C([0,1], M_{mn})$ for $m, n > 1$ denote the dimension-drop algebra given by the subalgebra of $C([0,1], M_m \otimes M_n)$ of continuous functions $f\colon [0,1] \to M_m \otimes M_n$ with $f(0) \in M_m \otimes 1_n$ and $f(1) \in 1_m \otimes M_n$. In the following we use only that the *Jian–Su algebra* $\mathcal{Z}$ (cf.[18]) is a simple unital C^*-algebra, that $\mathcal{Z}$ is an inductive limit of $\mathcal{I}(m_k, n_k)$ with $\min(m_k, n_k) \to \infty$ for $k \to \infty$, that $\mathcal{Z}$ does not contain a non-trivial projection, and that $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \otimes \cdots$.

Lemma 3.11 $\operatorname{cov}(\mathcal{I}(m,n), \min(n,m)) \le 2$ *and* $\operatorname{cov}(\mathcal{Z}) = 2$*.*

The proof follows from Proposition 3.5 and parts (2), (4) and (5) of Remark 3.3, because $\operatorname{dr}(\mathcal{I}(m,n)) = 1$. But we give an independent proof.

Proof. Let $a \in C([0,1], M_{mn})_+$ given by $a(t) := t1_{mn}$. Then $a \in \mathcal{I}(m,n)$, $a^{1/3}$ is n-homogenous and $(1-a)^{1/3}$ is m-homogenous in $\mathcal{I}(m,n)$. $1 = d_1^* a^{1/3} d_1 + d_2^* (1-a)^{1/3} d_2$ for $d_1 = a^{1/3}$ and $d_2 = (1-a)^{1/3}$. Hence, $\operatorname{cov}(\mathcal{I}(m,n), \min(n,m)) \le 2$.

For $k \in \mathbb{N}$ there are $n, m \ge k$ such that there is a unital *-morphism from $\mathcal{I}(m,n)$ into $\mathcal{Z}$. Thus, $\operatorname{cov}(\mathcal{Z}, k) \le 2$ by Remark 3.3(2).

$\operatorname{cov}(\mathcal{Z}, 2) > 1$ by Remark 3.3(5), because $1_{\mathcal{Z}}$ is finite and does not contain a non-trivial projection. Hence $\operatorname{cov}(\mathcal{Z}, k) = 2$ for $k = 2, 3, \ldots$. □

Corollary 3.12 *$A \otimes \mathcal{Z}$ is strongly purely infinite if every lower semi-continuous 2-quasi-trace on A_+ is trivial.*

Proof. Then every l.s.c. 2-quasi-trace $(A \otimes \mathcal{Z})_+ \to [0, \infty]$ is trivial. $F(A \otimes \mathcal{Z})$ contains a copy of $\mathcal{Z}$ unitally, because $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \otimes \cdots$. The result follows from Lemma 3.11, Remark 3.3(2) and Proposition 3.7. □

Corollary 3.13 *If A is simple and separable, and is neither stably finite nor purely infinite, then there is $k_0 < \infty$ such that, for all $m, n \geq k_0$, there is no unital *-morphism from $\mathcal{I}(m, n)$ into $F(A)$.*

Proof. The assumptions imply that every l.s.c. 2-quasi-trace on A_+ is trivial. Since A is simple and is not purely infinite, A is not weakly purely infinite. Thus $\mathrm{cov}(F(A)) = \infty$ by Proposition 3.7.

Let $k_0 := \inf\{k \in \mathbb{N};\ \mathrm{cov}(F(A), k) > 2\}$. If $h\colon \mathcal{I}(m, n) \to F(A)$ is a unital *-morphism, then $\mathrm{cov}(F(A), \min(m, m)) \leq \mathrm{cov}(\mathcal{I}(m, n), \min(m, n)) \leq 2$ by Remark 3.3(2) and Lemma 3.11. Thus $\min(m, n) < k_0$. □

Corollary 3.14 *Let $\mathcal{R}$ an example of a simple separable unital nuclear C^*-algebra that is neither stably finite nor purely infinite (cf. Rørdam [30]). Then $\mathrm{cov}(F(\mathcal{R})) = \infty$, $F(\mathcal{R})$ is stably finite, and $F(\mathcal{R})$ does not contain a simple unital C^*-subalgebra $B \neq \mathbb{C} \cdot 1$.*

Proof. $\mathrm{cov}(F(\mathcal{R})) = \infty$ by Proposition 3.7. $F(\mathcal{R})$ must be stably finite by Remark 2.13, because $\mathcal{R}$ is not (locally) purely infinite. There is no unital C^*-subalgebra $B \neq \mathbb{C} \cdot 1$ of $F(\mathcal{R})$, such that $B \otimes \mathcal{R}$ is weakly purely infinite (i.e. n–purely infinite for some n), because otherwise $a \otimes 1_n$ is properly infinite in $\rho(B \otimes \mathcal{R}) \otimes M_n \subset \mathcal{R}_\omega \otimes M_n$, for every $a \in \mathcal{R}$, which implies that $\mathcal{R}$ is n–purely infinite, a contradiction. Suppose that $B \neq \mathbb{C} \cdot 1$ is a simple C^*-subalgebra of $F(\mathcal{R})$ then there is also an antiliminal simple algebra B unitally contained in $F(\mathcal{R})$ (cf. Corollary 1.13). But then $B \otimes \mathcal{R}$ is purely infinite by [6, cor. 3.11], □

Question 3.15 *Does $F(\mathcal{R})$ contain a strictly antiliminal unital C^*-subalgebra B?*

A *positive* answer to Question 3.15 would show that:

(1) there exists a separable strictly antiliminal stably finite unital C^*-algebra that does not contain a non-trivial simple C^*-algebra unitally (because of 3.14 and because every strictly antiliminal unital C^*-algebra is the inductive limit of its separable strictly antiliminal C^*-subalgebras), and
(2) there are locally purely infinite algebras that are not weakly purely infinite (because $B \otimes \mathcal{R}$ is not weakly p.i. by the argument in the proof of 3.14, but is locally p.i. by [6, cor. 3.9(iv)]).

Question 3.16 *Suppose that A is a simple stably projection-less separable C^*-algebra and that $M_2 \oplus M_3$ is unitally contained in $F(A)$. Is A approximately divisible?*

If $1_{F(A)} \in M_2 \oplus M_3 \subset F(A)$, then there is a unital *-morphism from the infinite tensor product

$$E := (M_2 \oplus M_3) \otimes (M_2 \oplus M_3) \otimes \cdots$$

into $F(A)$ by Corollary 1.13. E contains a simple AF–algebra D unitally (communicated by M. Rørdam, May 2004). Every simple unital AF–algebra contains a copy of $M_2 \oplus M_3$ unitally. Thus, the property in the question equivalently means that $F(A)$ contains a copy of a simple AF–algebra unitally. Every simple unital AF–algebra absorbs a copy of $\mathcal{Z}$, cf. [18]. It follows that $A \cong A \otimes \mathcal{Z}$ (cf. Section 4).

The estimate for $\mathrm{cov}(F(A), m)$ in Proposition 3.4 is not optimal, e.g. $\mathrm{cov}(F(M_{2^\infty}), m) = 1$ and $\mathrm{cov}(F(M_{2^k}), 2) = \infty$ for all $k \in \mathbb{N}$, because $F(M_{2^\infty})$ contains a copy of M_{2^∞} unitally and $F(M_{2^k}) = F(\mathbb{C}) = \mathbb{C}$.
Since $F(M_{2^k}, M_{2^{k+m}}) = M_{2^m}$, one gets better estimates if one considers in some case also also $\mathrm{cov}(F(A_{n_k}, A_{n_{k+1}}), m)$ for suitable $n_1 < n_2 < \cdots$.

4 Self-Absorbing Subalgebras of $F(A)$

Suppose that $1_{F(A)} \in D \subset F(A)$ is a simple separable and nuclear unital C^*-subalgebra of $F(A)$. Then $\mathcal{D} := D^{\otimes\infty} := D \otimes D \otimes \cdots$ is unitally contained in $F(A)$ by Corollary 1.13.

Here we are interested in the question, when this implies that there is an isomorphism ψ from A onto $A \otimes \mathcal{D}$, and when ψ can be found such that ψ is approximately unitarily equivalent to the morphism $a \in A \mapsto a \otimes 1 \in A \otimes \mathcal{D}$.

Definitions 4.1 *Let A and D C^*-algebras, where D is unital. We say that A is* D-absorbing *(in a strong sense) if there exists an isomorphism ψ from A onto $A \otimes D$ that is approximately unitarily equivalent to the morphism $a \mapsto a \otimes 1$ (by unitaries in $\mathcal{M}(A \otimes D)$). We call A* stably D-absorbing *if $\mathcal{K} \otimes A$ is D-absorbing.*

A unital C^*-algebra D is self-absorbing *if D is D-absorbing.*

D has approximately inner flip *if the flip automorphism of $D \otimes D$ is approximately inner.*

If there exists $A \neq \{0\}$ such that A is D-absorbing, then D is simple and nuclear (cf. Lemma 4.9). Conversely, if D is simple, separable, unital, and nuclear, then $\mathcal{O}_2$ is D-absorbing (by classification theory).

If A and D are separable, D simple, unital and nuclear and A is $\mathcal{D}$-absorbing, then $D^{\otimes\infty}$ is unitally contained in $F(A)$. (cf. Proposition 4.11). This property is not enough to ensure that A is D-absorbing, as the following remark shows (see Appendix C for details):

Remark 4.2 *The infinite tensor product $\mathcal{O}_n \otimes \mathcal{O}_n \otimes \cdots$ is unitally contained in $F(\mathcal{O}_n)$. But the maps $\eta_{1,\infty}\colon a \mapsto a \otimes 1 \otimes 1 \otimes \ldots$ and $\eta_{2,\infty}\colon a \mapsto 1 \otimes a \otimes$*

$1 \otimes \ldots$ from $\mathcal{O}_n$ into $\mathcal{O}_n \otimes \mathcal{O}_n \otimes \ldots$ $(i = 1, 2)$ are not approximately unitarily equivalent for $n \geq 3$. In particular, ${\mathcal{O}_n}^{\otimes\infty}$ is not self-absorbing, and the flip on $({\mathcal{O}_n}^{\otimes\infty}) \otimes ({\mathcal{O}_n}^{\otimes\infty})$ is not approximately inner.

Let us fix some notation for this section. If D is a unital, we let $\mathcal{D} := D^{\otimes\infty} := D \otimes D \otimes \ldots$ denote the infinite tensor product of D.

We define $\eta_{k,n} \colon D \to D^{\otimes n}$ for $n = 2, 3, \ldots, \infty$, $k = 1, 2, \ldots$ with $k \leq n$ by $\eta_{k,n}(a) = 1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \otimes \cdots \otimes 1$ for $a \in D$ with a on k-th position. We let $\eta_1 := \eta_{1,2}$ and $\eta_2 := \eta_{2,2}$.

The different behavior of D and $\mathcal{D}$ can be seen from the following.

Remarks 4.3 *Suppose that D is a simple separable unital nuclear C^*-algebra that contains a copy of $\mathcal{O}_2$ unitally. Then:*
(1) The morphisms η_1 and η_2 are approximately unitarily equivalent in $D \otimes D$ and $D^{\otimes\infty} \cong \mathcal{O}_2$.

An example with $D \ncong \mathcal{D}$ is $D := \mathcal{P}_\infty$ the unique p.i.s.u.n. algebra in the UCT class with $K_0(\mathcal{P}_\infty) = 0$ and $K_1(\mathcal{P}_\infty) \cong \mathbb{Z}$.
(2) The flip automorphism on $\mathcal{P}_\infty \otimes \mathcal{P}_\infty$ is not *approximately inner.*
(3) There exist simple nuclear C^*-algebras D that contains a copy of $\mathcal{O}_2$ unitally and are not *purely infinite (e.g. the examples of Rørdam [30] are stably isomorphic to those algebras).*

See Appendix C for more explanation.

It shows that infinite tensor products $\mathcal{D} = D^{\otimes\infty}$ considerably loose properties of D. $\mathcal{D}$ is stably finite or purely infinite by [6, cor. 3.11] for simple D.

Below we see that $D \cong \mathcal{D}$ and every unital *-endomorphism of $\mathcal{D}$ is approximately inner if and only if $\mathcal{D}$ is self-absorbing and separable. Therefore we use the notation $\mathcal{D}$ also for self-absorbing algebras.

This class of separable self-absorbing algebras could be of interest for a classification theory of (not necessarily purely infinite) separable nuclear C^*-algebras up to tensoring with $\mathcal{D}$:
The classification of separable stable strongly purely infinite nuclear algebras is a classification of all separable stable nuclear C^*-algebras modulo tensor product with $\mathcal{D} = \mathcal{O}_\infty$. The strongly purely infinite algebras are contained in the (possibly larger) class of algebras A with the property that $\mathcal{O}_\infty$ is unitally contained in $F(C, A)$ for every separable *nuclear* C^*-subalgebra C of A_ω.

We list some results on self-absorbing $\mathcal{D}$ in the UCT-class, and point out some open questions on self-absorbing $\mathcal{D}$ in the UCT class that have a tracial state.

Proposition 4.4 *Let $\mathcal{D}$ a unital separable self-absorbing C^*-algebra. Then:*

(1) $\mathcal{D}$ is simple and nuclear. Either $\mathcal{D}$ is purely infinite or $\mathcal{D}$ has a unique tracial state.
(2) $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D} \otimes \cdots$.

(3) *Every unital endomorphism of $\mathcal{D}$ is approximately inner by unitaries in the* commutator *subgroup of* $\mathcal{U}(\mathcal{D})$.
(4) *If B is separable and $\mathcal{M}(B)$ contains a copy of $\mathcal{O}_2$ unitally, then B is $\mathcal{D}$-absorbing if and only if $F(B)$ contains a copy of $\mathcal{D}$ unitally.*
In particular, a separable algebra A is stably *$\mathcal{D}$-absorbing if and only if $\mathcal{D} \subset F(A)$.*
(5) *If the commutator subgroup of $\mathcal{U}(\mathcal{D})$ is contained in the connected component $\mathcal{U}_0(\mathcal{D})$ of 1, then every* stably *$\mathcal{D}$-absorbing separable C^*-algebra is $\mathcal{D}$-absorbing.*

It is a consequence of the basic Proposition 4.11 and of Corollaries 4.12 and 4.13. See end of this section for a proof.

We use the invariant $F(A)$ to give an alternative approach to permanence properties of the class of (strongly) $\mathcal{D}$-absorbing separable C^*-algebras, as e.g. studied by A. Toms and W. Winter [32], and we give a simple *necessary and sufficient* condition under which the class is closed under extensions (and is then automatically closed under Morita equivalence).

Theorem 4.5 *Suppose that $\mathcal{D}$ is unital, separable and self-absorbing.*

(1) *If B is a unital separable C^*-algebra and a copy of $\mathcal{D}$ is unitally contained in B_ω, then $B \otimes B \otimes \cdots$ is $\mathcal{D}$-absorbing.*
In particular:

$$\mathcal{D} \otimes M_2 \otimes M_3 \otimes \cdots \cong M_2 \otimes M_3 \otimes \cdots$$

if $\mathcal{D}$ is quasi-diagonal.
*If for every $n \in \mathbb{N}$ there exist $p, q \geq n$ and a unital *-morphism from $\mathcal{E}(M_p, M_q)$ into $\mathcal{D}_\omega$, then $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{Z}$.*
(2) *The class of* stably *$\mathcal{D}$-absorbing separable C^*-algebras is closed under inductive limits, passage to hereditary C^*-subalgebras, and to quotients.*
A unital separable algebra B is $\mathcal{D}$-absorbing if B is stably $\mathcal{D}$-absorbing.
(3) *The class of stably $\mathcal{D}$-absorbing separable algebras is closed under extensions, if and only if,*

$$\mathcal{E}(\mathcal{D}, \mathcal{D}) \cong \mathcal{E}(\mathcal{D}, \mathcal{D}) \otimes \mathcal{D}\,,$$

if and only if, the commutator subgroup of $\mathcal{U}(\mathcal{D})$ is in the connected component $\mathcal{U}_0(\mathcal{D})$ of 1.
(4) *If the class of stably $\mathcal{D}$-absorbing separable algebras is closed under extension, then every stably $\mathcal{D}$-absorbing algebra is $\mathcal{D}$-absorbing.*

We give a proof at the end of this section.

Parts (2)–(4) together imply that the class of $\mathcal{D}$-absorbing separable algebras is closed under all above considered operations, if and only if, $uvu^{-1}v \in \mathcal{U}_0(\mathcal{D})$ for all unitaries $u, v \in \mathcal{D}$. The property $[\mathcal{U}(\mathcal{D}), \mathcal{U}(\mathcal{D})] \subset \mathcal{U}_0(\mathcal{D})$ holds for simple purely infinite algebras $\mathcal{D}$, because the natural group morphism

$\mathcal{U}(\mathcal{D})/\mathcal{U}_0(\mathcal{D}) \to K_1(\mathcal{D})$ is an isomorphism (J. Cuntz [10]) if $\mathcal{D}$ is simple and purely infinite. A. Toms and W. Winter [32] obtained permanence results for tensorially $\mathcal{D}$–absorbing algebras under the (perhaps stronger) assumption that $\mathcal{U}(\mathcal{D})/\mathcal{U}_0(\mathcal{D}) \cong K_1(\mathcal{D})$ for self-absorbing $\mathcal{D}$.

Remark 4.6 *The Cuntz algebras $\mathcal{O}_2$, $\mathcal{O}_\infty$, the UHF algebras M_{p^∞} (p prime), the Jiang-Su algebra $\mathcal{Z}$ and all (finite and infinite) tensor products $\mathcal{D}_1 \otimes \mathcal{D}_2 \otimes \ldots$ are examples of self-absorbing $\mathcal{D}$ in the sense of Definition 4.1.*

Up to tensoring with $\mathcal{O}_\infty$ this list exhausts all $\mathcal{D}$ in the UCT class.

The Elliott invariants of this algebras exhaust all possible Elliott invariants of $\mathcal{D} \otimes \mathcal{Z}$ for self-absorbing $\mathcal{D}$ in the KTP class ($\supset$ UCT class).

They all have connected unitary groups, thus the class of separable $\mathcal{D}$-absorbing algebras is closed under inductive limits, extensions, passage to hereditary subalgebras and under passage to quotients.

The flip automorphism on $\mathcal{D} \otimes \mathcal{D}$ is (unitarily) homotopic to the identity for UHF-algebras $\mathcal{D}$, $\mathcal{D} = \mathcal{O}_2$ and $\mathcal{D} = \mathcal{O}_\infty$.

Clearly, M_{p^∞} has the required properties. The considered properties are invariant under infinite tensor products. $\mathcal{Z}$ has the properties by [18]. The others follow from KTP, UCT and the classification of p.i.s.u.n. algebras by means of KK-theory (see Appendix C for details, or [32] for an alternative proof). We do not know if η_1, η_2 are (unitarily) homotopic for $\mathcal{D} = \mathcal{Z}$.

The case of UCT algebras suggests:

Conjecture 4.7 *If $\mathcal{D}$ is self-absorbing and $\mathcal{D} \neq \mathcal{O}_2$ then*

$$\mathcal{D} \otimes \mathcal{O}_\infty \otimes M_2 \otimes M_3 \otimes \cdots \cong \mathcal{O}_\infty \otimes M_2 \otimes M_3 \otimes \cdots .$$

Recall from Proposition 1.9(4,5,9), that the natural *-morphism from the normalizer $\mathcal{N}(D_B) \subset \mathcal{M}(B)_\omega$ of $D_B \subset B_\omega$ into $\mathcal{M}(D_B)$ defines isomorphisms $\mathcal{N}(D_B)/\mathrm{Ann}(B, \mathcal{M}(B)_\omega) \cong \mathcal{M}(D_B)$ and

$$F(B) = (B' \cap \mathcal{M}(B)_\omega)/\mathrm{Ann}(B, M(B)_\omega) \cong B' \cap \mathcal{M}(D_B)$$

if B is σ–unital. It allows to improve [25, prop. 8.1] as follows:

Proposition 4.8 *Suppose that B is a separable C^*-algebra and A is a non-degenerate C^*-subalgebra of B. Let $\mathcal{U}_1 \subset \mathcal{M}(D_B)$ denote the image of the unitary group of $\mathcal{N}(D_B)$ in $\mathcal{M}(D_B)$.*

If there are unitaries $W_1, W_2, \ldots \in \mathcal{U}_1$ with $\lim_{n\to\infty} \|W_n a - a W_n\| = 0$ for every $a \in A$ and $\lim_{n\to\infty} \mathrm{dist}(W_n^* b W_n, A_\omega) = 0$ for every $b \in B$, then there is a unitary $U = \pi_\omega(u_1, u_2, \ldots) \in \mathcal{M}(B)_\omega$ with $U^* B U = A$.

*The *-isomorphism $\psi(a) := U a U^*$ from A onto B is approximately unitarily equivalent to the inclusion map $A \subset B$ by the unitaries $u_1^*, u_2^*, \ldots \in \mathcal{M}(B)$.*

If one can find the W_n even in $\mathcal{U}_0(\mathcal{M}(D_B))$ then $u_1, u_2, \ldots$ can be chosen in $\mathcal{U}_0(\mathcal{M}(B))$.

Proof. Let $G \subset \mathcal{U}(\mathcal{M}(B))$ a (countable) subgroup such that for each $n \in \mathbb{N}$ there is a sequence $(g_1, g_2, \ldots) \in G$ with $\pi_\omega(g_1, g_2, \ldots) \in \mathcal{N}(D_B) \subset \mathcal{M}(B)_\omega$ and

$$\pi_\omega(g_1, g_2, \ldots) + \mathrm{Ann}(B, \mathcal{M}(B)_\omega) = W_n \,.$$

Note that G can be found in $\mathcal{U}_0(\mathcal{M}(B))$ if $W_n \in \mathcal{U}_0(\mathcal{M}(B))$, because unitaries in

$$\mathcal{U}_0(\mathcal{M}(D_B)) \cong \mathcal{U}_0(\mathcal{N}(D_B, \mathcal{M}(B)_\omega)/\mathrm{Ann}(B, \mathcal{M}(B)_\omega))$$

lift to unitaries in $\mathcal{U}_0(\mathcal{M}(B)_\omega)$ and $\mathcal{U}_0(\mathcal{M}(B)_\omega) \subset (\mathcal{U}_0(\mathcal{M}(B)))_\omega$.

Let $(a_1, a_2, \cdots)$ and $(b_1, b_2, \ldots)$ dense sequences in the unit-ball of A respectively of B. Consider the sequence of functions $f_1, f_2, \ldots$ on G given by $f_{2k-1}(g) := \|ga_k - a_k g\|$ and $f_{2k}(g) := \mathrm{dist}(g^{-1}b_k g, A)$ for $k \in \mathbb{N}$. Then $G \subset \mathcal{M}(B)$ and $(f_1, f_2, \ldots)$ satisfy the assumptions of Remark A.2: indeed, use the representing sequences for W_n and apply the assumptions on W_n. It follows, that there is a sequence $(v_1, v_2, \ldots)$ in $G \subset \mathcal{U}(\mathcal{M}(B))$ such that $\lim_n f_k(v_n) = 0$ for all $k \in \mathbb{N}$. This means that $\mathrm{id}\colon A \hookrightarrow B$ and $(v_1, v_2, \ldots)$ satisfy the assumptions of [25, prop. 8.1]. The proof of [25, prop. 8.1] shows that there is a sequence of unitaries $u_1, u_2, \ldots \in G$, such that $U := \pi_\omega(u_1, u_2, \ldots)$ is as required. □

Lemma 4.9 *Suppose that D and E are C^*-algebras $a \in D_+$, that $h\colon C_0((0,1], D) \to E$ is a *-morphism with $h(f_0 \otimes a) \neq 0$. If there is a net $\{U_\tau\}$ unitaries in $\mathcal{M}(E \otimes D)$ such that $\{U_\tau^*(h(f_0 \otimes d) \otimes a)U_\tau\}$ converges to $h(f_0 \otimes a) \otimes d$ for all $d \in D$, then:*

(1) D is simple and nuclear.
(2) If there are $\delta > 0$ and a lower semi-continuous 2-quasi-trace $\mu\colon E_+ \to [0, \infty]$ with $0 < \mu(h((f_0 - \delta)_+ \otimes (a - \delta)_+)) < \infty$, then all l.s.c. 2-traces $\nu\colon D_+ \to [0, \infty]$ are additive and are proportional to the trace

$$a \in D_+ \mapsto \mu(h((f_0 - \delta)_+ \otimes a)) \,.$$

Proof. (1): Use inner automorphisms of $E \otimes D$ composed with slice maps from $E \otimes D$ into D.

(2): Since D is simple and nuclear, every l.s.c. 2-quasi-trace ν on D_+ is an additive trace, and there is an extended l.s.c. 2-quasi-trace $\lambda\colon (E \otimes D)_+ \to [0, \infty]$ with $\lambda(e \otimes d) = \mu(e)\nu(d)$ for $d \in D_+$, $e \in E_+$ with $\nu(d) < \infty$ and $\mu(e) < \infty$, cf. [6, rem. 2.29, proof of cor. 3.11(iv)]. ν is semi-finite and faithful if ν is non-trivial, in particular $0 < \nu((a - \delta)_+) < \infty$ for $\delta > 0$.

Then $\mu(h((f \otimes d))\nu(b) \leq \mu(h((f \otimes b))\nu(d)$ for $d \in D_+$, $f := (f_0 - \delta)_+$ and $b := (a - \delta)_+$, because $h(f \otimes d) \otimes b$ is the limit of $U_\tau(h(f \otimes b) \otimes d)U_\tau^*$ and λ is l.s.c. A similar argument shows "$\geq$". Thus $\nu(d) = \gamma\mu(h(f \otimes d))$ for all $d \in D_+$, where $\gamma := \nu(b)/\mu(h(f \otimes b))$. □

Remark 4.10 *Suppose that A and D are separable where D is unital. Consider the following conditions for A and D:*

*(β) The two *-morphisms* $\mathrm{id}_A \otimes \eta_1$ *and* $\mathrm{id}_A \otimes \eta_2$ *from* $A \otimes D$ *into* $A \otimes (D \otimes D)$ *are approximately unitarily equivalent by unitaries in the connected component* $\mathcal{U}_0(\mathcal{M}(A \otimes D \otimes D))$ *of the unitaries in* $\mathcal{M}(A \otimes D \otimes D)$.

*(β') The *-morphisms* $\mathrm{id}_A \otimes \eta_{1,\infty}$ *and* $\mathrm{id}_A \otimes \eta_{2,\infty}$ *from* $A \otimes D$ *into* $A \otimes (D \otimes D \otimes \cdots)$ *are approximately unitarily equivalent by unitaries in* $\mathcal{U}_0(\mathcal{M}(A \otimes D \otimes D \otimes \cdots))$.

Then:

(1) (β) implies (β'), (β') implies that D *is simple and nuclear, and that for every unital endomorphism* φ *of* $D \otimes D \otimes \cdots$ *the endomorphism* $\mathrm{id}_A \otimes \varphi$ *of* $A \otimes D \otimes D \otimes \ldots$ *is approximately inner by unitaries in* $\mathcal{U}_0(A \otimes D \otimes D \otimes \cdots)$. *In particular ($\beta$) holds with* $D \otimes D \otimes \ldots$ *in place of* D.

(2) If A_+ *has a non-trivial lower semi-continuous (extended)* 2*-quasi-trace, then (β') implies that* D *has a unique tracial state.*

(3) The condition (β') is satisfied if the morphisms $\eta_{1,\infty}$ *and* $\eta_{2,\infty}$ *from* D *into* $D \otimes D \otimes \cdots$ *are approximately unitarily equivalent and if* $\mathcal{M}(A)$ *contains a copy of* $\mathcal{O}_2$ *unitally (e.g. if* A *is stable).*

(4) The condition (β') is satisfied for every A *if the morphisms* $\eta_{1,\infty}$ *and* $\eta_{2,\infty}$ *from* $D \to D \otimes D \otimes \cdots$ *are approximately unitarily equivalent by unitaries in the connected component* $\mathcal{U}_0(D \otimes D \otimes \cdots)$ *of* 1 *in* $\mathcal{U}_0(D \otimes D \otimes \cdots)$.

(5) If $A \cong A \otimes D \otimes D \otimes \cdots$, *then ($\beta$') implies ($\beta$).*

The morphisms η_k and $\eta_{k,\infty}$ are above defined. Recall that $\mathcal{U}(A \otimes D \otimes D \otimes \ldots)$ is connected in norm-topology if A is stable and σ-unital (by a result of J. Cuntz and N. Higson).

Proof. (2) follows from from Lemma 4.9(2).

(4) is obvious.

(3): Since $\eta_{1,\infty}$ and $\eta_{2,\infty}$ are approximately unitarily equivalent, we get from Lemma 4.9(1) that $\mathcal{D}$ is simple and nuclear.

By classification theory, $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{D}$, because $\mathcal{D}$ is simple, separable and nuclear. The group of unitaries $\mathcal{U}(\mathcal{O}_2)$ is connected (cf. [10]).

(1): It is obvious that (β) implies (β'). D is simple and nuclear by Lemma 4.9(1).

Let B_1 and B_2 unital algebras, and psi_1, ψ_2 unital morphisms from B_1 into B_2. We use the notation $\psi_1 \sim \psi_2$ if $\mathrm{id}_A \otimes \psi_1$ and $\mathrm{id}_A \otimes \psi_2$ are approximately unitarily equivalent by unitaries in $\mathcal{U}_0(\mathcal{M}(A \otimes B_2))$.

There are obvious composition rules:
$\psi_1 \sim \psi_2$ and $\psi_2 \sim \psi_3$ imply $\psi_1 \sim \psi_3$. If $\lambda\colon B_2 \to B_3$ and $\mu\colon B_0 \to B_1$ are unital, and if $\psi_1 \sim \psi_2$, then $\lambda \circ \psi_1 \sim \lambda \circ \psi_2$ and $\psi_1 \circ \mu \sim \psi_2 \circ \mu$.

For $n \in \mathbb{N}$ and permutations σ of $\{1, \ldots, n\}$, we define *-morphisms

$$\psi_\sigma\colon D^{\otimes n} \to D^{\otimes n} \otimes 1 \otimes 1 \otimes \ldots \subset \mathcal{D}$$

by

$$\psi_\sigma(d_1 \otimes d_2 \otimes \ldots \otimes d_n) = d_{\sigma(1)} \otimes d_{\sigma(2)} \otimes \ldots \otimes d_{\sigma(n)} \otimes 1 \otimes 1 \otimes \cdots .$$

Further let $\epsilon_n := \psi_\sigma$ for $\sigma = \mathrm{id}$ of $\{1, \ldots, n\}$. For $m < n$ we define $\nu_{m,n} \colon D^{\otimes m} \to D^{\otimes n}$ by $\epsilon_m = \epsilon_n \circ \mu_{m,n}$, i.e.

$$\nu_{m,n}(d_1 \otimes \cdots \otimes d_m) = d_1 \otimes \cdots \otimes d_m \otimes 1 \otimes \cdots \otimes 1 \,.$$

The condition (β') implies that $\psi_\sigma \sim \epsilon_n$ for every transposition σ. Since every permutation is a product of transpositions, one can see by the rules for the relation $\sim$ that $\psi_\sigma \sim \epsilon_n$.

Let τ_1 and τ_2 denote the endomorphisms of $\mathcal{D}$ given by

$$\tau_1(d_1 \otimes d_2 \otimes \cdots \otimes d_n \otimes \cdots) = d_1 \otimes 1 \otimes d_2 \otimes 1 \otimes \cdots \otimes 1 \otimes d_n \otimes 1 \otimes \cdots$$

respectively

$$\tau_2(d_1 \otimes d_2 \otimes \cdots \otimes d_n \otimes \cdots) = 1 \otimes d_1 \otimes 1 \otimes d_2 \otimes \cdots \otimes 1 \otimes d_n \otimes 1 \otimes \cdots$$

Since there is a permutation σ of $\{1, \ldots, 2n\}$ with $\tau_\sigma \circ \epsilon_n = \psi_\sigma \circ \nu_{n,2n}$, we get that ($\beta$') implies that $\tau_k \circ \epsilon_n \sim \epsilon_n$ for $k = 1, 2$, $n \in \mathbb{N}$. It follows that

$$\tau_1 \sim \mathrm{id}_{\mathcal{D}} \sim \tau_2 \,.$$

We denote by γ the isomorphism from $\mathcal{D}$ onto $\mathcal{D} \otimes \mathcal{D}$ onto $\mathcal{D}$ with

$$\gamma\left((d_1 \otimes d_2 \otimes \cdots) \otimes (e_1 \otimes e_2 \otimes \ldots)\right) = (d_1 \otimes e_1 \otimes d_2 \otimes e_2 \otimes \cdots)$$

for $d_1, e_1, d_2, e_2, \ldots \in D$.

Then $\gamma \circ \eta_k = \tau_k \sim \mathrm{id}$ for $k = 1, 2$. It follows $\eta_1 = \gamma^{-1} \circ \tau_1 \sim \gamma^{-1} \circ \tau_2 = \eta_2$

Let $\psi \colon \mathcal{D} \to \mathcal{D}$ unital. Then

$$\psi \sim \gamma\eta_1\psi = \gamma(\psi \otimes \mathrm{id})\eta_1 \sim \gamma(\psi \otimes \mathrm{id})\eta_2 = \gamma\eta_2 = \tau_2 \sim \mathrm{id} \,.$$

(5): Conditions (β) and (β') are preserved if one passes over to isomorphic algebras, e.g. if $E \cong D$, then A and E satisfy (β), with E in place of D, if and only if, A and D satisfy (β).

Let $B := D \otimes D$, $\mathcal{D} := D \otimes D \otimes \cdots$, and let $\tau \colon B \to B$ denote the flip map $\tau \colon b_1 \otimes b_2 \mapsto b_2 \otimes b_1$.

Suppose that A and D satisfy condition (β'). Then A and $\mathcal{D} \otimes B \cong \mathcal{D}$ satisfy by part (1), that for every isomorphism φ of $\mathcal{D} \otimes B$ the isomorphism $\mathrm{id}_A \otimes \varphi$ of $A \otimes \mathcal{D} \otimes B$ is approximately unitarily equivalent to $\mathrm{id} = \mathrm{id}_A \otimes \mathrm{id}_{\mathcal{D}} \otimes \mathrm{id}_B$ by unitaries in $\mathcal{U}_0(A \otimes \mathcal{D} \otimes B)$ This applies to $\varphi := \mathrm{id}_{\mathcal{D}} \otimes \tau$.

If there is an isomorphism λ from A onto $A \otimes \mathcal{D}$ then there is a unital morphism Ψ from $\mathcal{M}(A \otimes (\mathcal{D} \otimes B))$ onto $\mathcal{M}(A \otimes B)$ with $\Psi(A \otimes (\mathcal{D} \otimes 1_B)) = A \otimes 1_B$ and $\Psi(a \otimes d \otimes b) = \lambda(a \otimes d) \otimes b$ for $a \in A$, $d \in \mathcal{D}$ and $b \in B$. It follows, that $\mathrm{id}_A \otimes \tau = \Psi \circ (\mathrm{id}_A \otimes \mathrm{id}_{\mathcal{D}} \otimes \tau) \circ \Psi^{-1}$ is approximately unitarily equivalent to $\mathrm{id}_A \otimes \mathrm{id}_B$ by unitaries in $\mathcal{U}_0(\mathcal{M}(A \otimes B))$.

In particular, A and D satisfy condition (β), because $\tau \circ \eta_1 = \eta_2$. □

The following proposition is the basic observation of this section. It generalizes [25, thm. 8.2] and observations of Effros and Rosenberg [15]. The proof uses Proposition 4.8. Here we consider a property that is a bit stronger than D-absorption.

Proposition 4.11 *Suppose A and D are separable, and that D is unital. Then the following are equivalent:*

(1) There is an isomorphism φ from A onto $A \otimes D$ that is approximately unitarily equivalent to $a \in A \mapsto a \otimes 1$ by unitaries in $\mathcal{U}_0(A \otimes D)$.
(2) A and D satisfy condition (β) of Remark 4.10 and $F(A)$ contains a copy of D unitally.
(3) A and D satisfy condition (β') of Remark 4.10 and $F(A)$ contains a copy of D unitally.
(4) There is an isomorphism ψ from A onto $A \otimes D \otimes D \otimes \cdots$ that is approximately unitarily equivalent to $a \mapsto a \otimes 1$ by unitaries in $\mathcal{U}_0(\mathcal{M}(A \otimes D \otimes D \otimes \cdots))$.
(5) A and D satisfy (β') and $A \cong A \otimes D \otimes D \otimes \cdots$.

In part (5) we don't suppose that the isomorphism from A onto $A \otimes \mathcal{D}$ is approximately unitarily equivalent to $a \mapsto a \otimes 1 \otimes 1 \otimes \cdots$.

Proof. (1)⇒(2): Let $\varphi\colon A \to A \otimes D$ as in part (1). Then $a \mapsto \varphi(a) \otimes 1$ is approximately unitarily equivalent to $a \mapsto a \otimes 1 \otimes 1$ by unitaries in $\mathcal{U}_0(A \otimes D \otimes D)$. The same must happen for $a \mapsto (\mathrm{id}_A \otimes \sigma)(\varphi(a) \otimes 1)$, because $\mathrm{id}_A \otimes \sigma$ extends to an automorphism of $\mathcal{M}(A \otimes D \otimes D)$. If we let $a := \varphi^{-1}(f)$ for $f \in A \otimes D$, then this shows that $f \to (\mathrm{id}_A \otimes \sigma)(f \otimes 1)$ and $f \to f \otimes 1$ are approximately unitarily equivalent, In particular, A and D satisfy condition (β) of Remark 4.10, and D is simple and nuclear by Remark 4.10(1).

The non-degenerate endomorphism $a \mapsto \varphi^{-1}(a \otimes 1)$ of A is approximately unitarily equivalent to id_A. If $u_1, u_2, \ldots \in \mathcal{M}(A)$ is a sequence of unitaries with $\lim u_n^* \varphi^{-1}(a \otimes 1) u_n = a$ for $a \in A$, then

$$\varphi_n\colon\ d \in D \to u_n^* \mathcal{M}(\varphi^{-1})(1 \otimes d) u_n \in \mathcal{M}(A)$$

is a unital *-monomorphism with $\lim \|[\varphi_n(d), a]\| = 0$, i.e.

$$\pi_\omega(\varphi_1(d), \varphi_2(d), \ldots) \in (A, \mathcal{M}(A))^c\,.$$

Since $F(A) \cong (A, \mathcal{M}(A))^c / \mathrm{Ann}(A, \mathcal{M}(A)_\omega)$ and D is simple, it follows that $F(A)$ contains a copy of D unitally.

(2)⇒(3): is obvious.

(5)⇒(3): Property (β') implies that D is *simple* and nuclear, cf. Remark 4.10(1). If λ is an isomorphism from $A \otimes D \otimes D \otimes \ldots$ onto A, then λ extends to a unital *-isomorphism from $\mathcal{M}(A \otimes D \otimes D \otimes \ldots)$ onto $\mathcal{M}(A)$. For $d \in D$ let

$$\varphi_n(d) := \lambda(1_{\mathcal{M}(A)} \otimes 1 \otimes \cdots \otimes 1 \otimes d \otimes 1 \otimes \cdots) \in \mathcal{M}(A \otimes D \otimes D \otimes \cdots)$$

with d on n-th position. This defines unital *-morphisms from D into $\mathcal{M}(A)$ with $\lim \|[\varphi_n(d), a]\| = 0$. Now deduce (3) as in the proof of the implication (1)⇒(2).

(3)⇒(4): By Remark 4.10(1), D must be simple and nuclear, and condition (β) is satisfied for A and $\mathcal{D} := D \otimes D \otimes \cdots$ (in place of D).

By Corollary 1.13, there is also a copy of $\mathcal{D} := D \otimes D \otimes \cdots$ unitally contained in $F(A)$, because A and D are separable, and D is unital, simple and nuclear.

Let $A \subset B := A \otimes \mathcal{D}$ (and identify A with $A \otimes 1_{\mathcal{D}}$). We show that A and B satisfy the assumptions of Proposition 4.8:

Let $h\colon \mathcal{D} \to F(A)$ a unital *-morphism. There is an isomorphism λ from $A \otimes \mathcal{D} \otimes \mathcal{D}$ into B_ω with $\lambda(a \otimes 1 \otimes 1) = a \in A_\omega \subset B_\omega$, $\lambda(a \otimes d \otimes 1) = \rho_A(h(d) \otimes a) \in D_A \subset A_\omega$, and $\lambda(a \otimes 1 \otimes d) = a \otimes d \in B$. λ is give by application of

$$(\rho_A \circ \sigma) \otimes \mathrm{id}_{\mathcal{D}}\colon\ A \otimes^{max} F(A) \otimes \mathcal{D} \to A_\omega \otimes \mathcal{D} \subset B_\omega$$

on $\mathrm{id}_A \otimes h \otimes \mathrm{id}_{\mathcal{D}}$. (Here σ means the flip isomorphism $a \otimes b \mapsto b \otimes a$).

I.e. $A \otimes \mathcal{D} \otimes \mathcal{D}$ may be considered as a non-degenerate C^*-subalgebra of $\overline{A(B_\omega)A} = D_B$.

The image of λ is a non-degenerate subalgebra of D_B. Thus

$$\mathcal{M}(\lambda)\colon\ \mathcal{M}(A \otimes \mathcal{D} \otimes \mathcal{D}) \to \mathcal{M}(D_B)$$

exists and is unital. Since A and $\mathcal{D}$ satisfy (β), we find a sequence of unitaries $W_n = \mathcal{M}(\lambda)(V_n) \in \mathcal{U}_0(\mathcal{M}(D_B))$ with $\lim_n W_n^* \lambda(a \otimes 1 \otimes d) W_n = \lambda(a \otimes d \otimes 1)$ for all $a \in A$ and $d \in D$. Thus $(W_1, W_2, \ldots)$ satisfies the assumptions of Proposition 4.8. It follows that there is an isomorphism ψ from A onto $B = A \otimes \mathcal{D}$ that is approximately inner by unitaries in $\mathcal{U}_0(A \otimes \mathcal{D})$, i.e. ψ is as stipulated in (3).

(4)⇒(1): If we apply the above verified implications (1)⇒(2) to A and $\mathcal{D} := D \otimes D \otimes \cdots$ in place of D, then we get that condition (β) is satisfied for A and $\mathcal{D}$. It follows that $\mathcal{D}$ is simple and nuclear.

By assumption, there is an isomorphism $\psi\colon\ A \to A \otimes \mathcal{D}$ from A onto $A \otimes \mathcal{D}$ such that $a \in A \mapsto a \otimes 1 \in A \otimes \mathcal{D}$ is approximately unitarily equivalent to ψ by unitaries in $\mathcal{U}_0(\mathcal{M}(A \otimes \mathcal{D}))$.

It follows that $a \in A \mapsto \psi^{-1}(a \otimes 1_{\mathcal{D}}) \in A$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{A}}$ by unitaries in $\mathcal{U}_0(\mathcal{M}(A))$. Let $\lambda\colon\ \mathcal{D} \to \mathcal{D} \otimes D$ the isomorphism given by $\lambda(d_1 \otimes d_2 \otimes \cdots) := (d_2 \otimes d_3 \otimes \cdots) \otimes d_1$. Then $\varphi := (\psi^{-1} \otimes \mathrm{id}_D) \circ (\mathrm{id}_A \otimes \lambda) \circ \psi$ is an isomorphism from A onto $A \otimes D$ and is approximately unitarily equivalent to $a \in A \mapsto \psi^{-1}(a \otimes 1_{\mathcal{D}}) \otimes 1_D \in A \otimes D$ by unitaries in $\mathcal{U}_0(\mathcal{M}(A \otimes D))$. Thus, the isomorphism $\varphi\colon\ A \to A \otimes D$ is approximately unitarily equivalent to $a \mapsto a \otimes 1$ by unitaries in $\mathcal{U}_0(A \otimes D)$.

(4)⇒(5): Since (4) implies (1), it implies also (2) and (3). Thus (4) implies condition (β') for A and D. $A \otimes A \otimes D \otimes D \otimes \ldots$ is part of (4). □

Corollary 4.12 *Suppose that D is unital and separable, and let $\mathcal{D} := D^{\otimes\infty} := D \otimes D \otimes \cdots$. Following properties (1)–(4) of D are equivalent:*

(1) Any two endomorphisms φ and ψ of $\mathcal{D}$ are approximately unitarily equivalent by commutators $u_n = v_n^* w_n^* v_n w_n$ *of unitaries v_n, w_n in $\mathcal{D}$.*
(2) The flip automorphism $\sigma\colon\ d \otimes e \mapsto e \otimes d$ of $\mathcal{D} \otimes \mathcal{D}$ is approximately inner,
(3) $\mathcal{D}$ is self-absorbing.
(4) The morphisms $\eta_{1,\infty}\colon\ d \mapsto d \otimes 1 \otimes 1 \otimes \cdots$ and $\eta_{2,\infty}\colon\ d \mapsto 1 \otimes d \otimes 1 \otimes \cdots$ from D into $\mathcal{D}$ are approximately unitarily equivalent in $\mathcal{D}$.

Proof. (1)⇒(2): Since $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D}$ by some isomorphism $\psi\colon\ \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$, we get that $\psi^{-1}\sigma\psi$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{D}}$. Thus, σ is an approximately inner automorphism of $\mathcal{D} \otimes \mathcal{D}$.

(2)⇒(3): $\mathcal{D}$ is simple and nuclear by Lemma 4.9(1). Let $A := \mathcal{K} \otimes \mathcal{D}$, then $A \cong A \otimes \mathcal{D} \otimes \mathcal{D} \otimes \cdots$ (by any isomorphism from $\mathcal{D}$ to $\mathcal{D} \otimes \mathcal{D} \otimes \cdots$).

Since $\eta_1 = \sigma \circ \eta_2$, we get that $\mathrm{id}_A \otimes \eta_1$ and $\mathrm{id}_A \otimes \eta_2$ are approximately unitarily equivalent by unitaries in $\mathcal{O}_2 \otimes \mathcal{D} \otimes \mathcal{D} \subset \mathcal{M}(A \otimes \mathcal{D} \otimes \mathcal{D})$. The unitary group of $\mathcal{O}_2 \otimes \mathcal{D} \otimes \mathcal{D} \cong \mathcal{O}_2$ is connected. Thus, A and $\mathcal{D}$ satisfy condition (β) (with $\mathcal{D}$ in place of D).

It follows that Proposition 4.11 can be applied on A and $\mathcal{D}$. It leads to an isomorphism ψ from A onto $A \otimes \mathcal{D}$ that is approximately unitarily equivalent to $a \mapsto a \otimes 1$. Since $\mathcal{D}$ is unital, ψ defines an isomorphism from $\mathcal{D} \cong e_{1,1} \otimes \mathcal{D}$ onto $\mathcal{D} \otimes \mathcal{D}$, that is approximately unitarily equivalent to $d \mapsto d \otimes 1$, i.e. $\mathcal{D}$ is self-absorbing.

(3)⇒(4): If $\mathcal{D} := D^{\otimes\infty}$ is self-absorbing, then $A := \mathcal{K} \otimes D$ and D satisfy part (4) of Proposition 4.11. Thus, A and D fulfill condition (β) by the implication (4)⇒(2) of 4.11. But this means that $\mathrm{id}_{\mathcal{D}} \otimes \eta_k\colon \mathcal{D} \otimes D \to \mathcal{D} \otimes (D \otimes D)$, with $k = 1, 2$ are approximately unitarily equivalent. The latter is an equivalent formulation of (4).

(4)⇒(1): $A := \mathcal{K}$ and D satisfy condition (β') of Remark 4.10. Thus, by part (1) of 4.10, $\mathrm{id}_{\mathcal{K}} \otimes \psi$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{K}} \otimes \mathrm{id}_{\mathcal{D}}$ for every unital endomorphism of $\mathcal{D} := D \otimes D \otimes \cdots$. This implies that any two unital endomorphisms of $\mathcal{D}$ are approximately unitarily equivalent.

It implies that $u \otimes u^* \otimes 1 \otimes \cdots \in \mathcal{U}(\mathcal{D})$ for $u \in \mathcal{U}(D^{\otimes n})$ is in the norm closure of the set of commutators $\{wv^*w^*v\,;\ v, w \in \mathcal{U}(\mathcal{D})\}$ in $\mathcal{U}(\mathcal{D})$. Indeed: the flip σ_n on $D^{\otimes n} \otimes D^{\otimes n}$ extends to an isomorphism λ of $\mathcal{D}$ with $\lambda(a \otimes b \otimes 1 \otimes \cdots) = a \otimes b \otimes 1 \otimes \cdots$ for $a, b \in D^{\otimes n}$. Since λ is approximately inner, we get a sequence of unitaries $v_n \in \mathcal{U}(\mathcal{D})$ with $u \otimes u^* \otimes 1 \otimes \cdots = w\lambda(w^*) = \lim_n wv_n^*w^*v_n$ for $w := u \otimes 1 \otimes 1 \otimes \cdots$.

If $X \subset \mathcal{D}$ is a finite subset of the contractions in $\mathcal{D}$ and if $v \in \mathcal{U}(\mathcal{D})$, then for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and $u \in \mathcal{U}(D^{\otimes n})$, such that $\|v^*dv - w^*dw\| < \varepsilon$ for $d \in X$ where $w := u \otimes u^* \otimes 1 \otimes \cdots$. Recall here that one can find n such that the elements of $X \cup \{v\}$ have distance $< \varepsilon/9$ from $D^{\otimes n} \otimes 1 \otimes \cdots \subset \mathcal{D}$.

It follows that unital endomorphisms φ and ψ of $\mathcal{D}$ are approximately unitarily equivalent by unitaries w_n in the set of commutators in $\mathcal{U}(\mathcal{D})$. □

Proposition 4.11 (with $A := \mathcal{K} \otimes D$) and Corollary 4.12 immediately imply the following corollary:

Corollary 4.13 *If D is a unital and separable, then D is self-absorbing (in the sense of Definitions 4.1) if and only if $D \cong D \otimes D \otimes \cdots$ and all endomorphisms of D are approximately unitarily equivalent by unitaries in the commutator subgroup of $\mathcal{U}(D)$.*

Proof. If $D \cong D \otimes D \otimes \cdots$ and all endomorphisms of $\mathcal{D} := D \otimes D \otimes \cdots$ are unitarily equivalent, then $\mathcal{D} \cong D$ is self-absorbing by Corollary 4.12(3). If D is self-absorbing, then the implication (1)⇒(4) of Proposition 4.11 applies to $A := \mathcal{K} \otimes D$ and D. Thus, there is an isomorphism ψ from $\mathcal{K} \otimes D$ onto $(\mathcal{K} \otimes D) \otimes \mathcal{D}$, such that ψ is approximately unitarily equivalent to $a \in \mathcal{K} \otimes D \mapsto a \otimes 1$. Since D is unital, this implies that $D \cong D \otimes D \otimes \ldots$. □

Corollary 4.14 *If A is separable and if there is a unital *-morphism from $M_2 \oplus M_3$ into $F(A)$ then $A \cong \mathcal{Z} \otimes A$.*

It could be that $1_{F(A)} \in M_2 \oplus M_3 \subset F(A)$ does *not* imply approximate divisibility of A in general, cf. Question 3.16.

Proof. Let $E := (M_2 \oplus M_3) \otimes (M_2 \oplus M_3) \otimes \cdots$. There is a sequence of unital *-morphisms h_n from $\mathcal{E}(M_{p_n}, M_{q_n})$ into E such that $\gcd(p_n, q_n) = 1$ and $p_n, q_n \geq n$. This defines a unital morphism from $\mathcal{Z}$ into E_ω. Since $\mathcal{Z}$ is self-absorbing, this implies $E \otimes \mathcal{Z} \cong E$ by Theorem 4.5(1). There is a unital *-morphism from E into $F(A)$ by Corollary 1.13. Thus $\mathcal{Z} \subset F(A)$ unitally. Since $\mathcal{U}(\mathcal{Z}) = \mathcal{U}_0(\mathcal{Z})$ and $\mathcal{Z}$ is tensorially self-absorbing, $A \cong A \otimes \mathcal{Z}$ by Proposition 4.4(4,5). □

Proof (of Proposition 4.4). (1,2,3): $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D} \otimes \cdots$ and every unital endomorphism of $\mathcal{D}$ is approximately inner by unitaries in the commutator group, cf. by Corollary 4.13. In particular, the flip automorphism of $\mathcal{D} \otimes \mathcal{D} \cong \mathcal{D}$ is approximately inner. Thus $\mathcal{D}$ is simple and nuclear and has at most one tracial state by Lemma 4.9. Since $\mathcal{D}$ is tensorially non-prime, it follows from [6, cor. 3.11(i)], that either $\mathcal{D}$ is purely infinite or $\mathcal{D}$ is stably finite. If a unital nuclear C^*-algebra $\mathcal{D}$ is stably finite then $\mathcal{D}$ admits tracial state (by results of B. Blackadar, J. Cuntz and U. Haagerup).

(3): See Corollary 4.13 or Corollary 4.12(1).

(4): The pair of algebras $(B, \mathcal{D})$ satisfies condition (β') by part (3) of Remark 4.10, because the flip automorphism on $\mathcal{D} \otimes \mathcal{D}$ is approximately inner by part (2) and Corollary 4.12(2). By the equivalences (1)⇔(3) of Proposition 4.11, B is $\mathcal{D}$-absorbing if and only if there is a copy of $\mathcal{D}$ unitally contained in $F(B)$

$\mathcal{M}(\mathcal{K} \otimes A)$ contains a unital copy of $\mathcal{O}_2$ for every A, and $F(A) \cong F(\mathcal{K} \otimes A)$ for separable A.

(5): By part (3), the maps η_2 and η_2 from $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D} \otimes \cdots$ into $\mathcal{D} \otimes \mathcal{D} \cong \mathcal{D}$ are approximately unitarily equivalent by unitaries in the commutator subgroup of $\mathcal{U}(\mathcal{D})$.

By assumption, the commutator subgroup is contained in $\mathcal{U}_0(\mathcal{D})$. Thus, by Remark 4.10(4), the pair of algebras $(A, \mathcal{D})$ satisfies condition (β') for every separable algebra A. Now Proposition 4.11 applies: A is $\mathcal{D}$-absorbing if and only if $F(A)$ contains a copy of $\mathcal{D}$ unitally. □

Proof (of Theorem 4.5). (1): Let $B^{\otimes\infty} := B \otimes B \otimes \cdots$ There is a unital *-morphism

$$\psi\colon\ B_\omega \to (B \otimes B \otimes \cdots)^c \cong F(B \otimes B \otimes \cdots)\,.$$

It is the ultrapower $\psi := (\psi_1, \psi_2, \ldots)_\omega$ of the morphisms $\psi_n\colon B \to B^{\otimes\infty}$ given by $\psi_n(b) := 1_n \otimes b \otimes 1_\infty$, where $1_{n+1} := 1_n \otimes 1$ and $1_\infty := 1 \otimes 1 \otimes \cdots$.

If $\varphi\colon \mathcal{D} \to B_\omega$ is unital, then $\psi \circ \varphi$ is a unital *-morphism from $\mathcal{D}$ into $F(B \otimes B \otimes \cdots)$. Since $\mathcal{D}$ is simple, a copy of $\mathcal{D}$ is unitally contained in $F(B \otimes B \otimes \cdots)$. Thus $B \otimes B \otimes \cdots$ is stably $\mathcal{D}$-absorbing by Proposition 4.4(4).

If $\mathcal{D}$ is quasi-diagonal, then $\mathcal{D}$ is unitally contained in B_ω for $B := M_2 \otimes M_3 \otimes \cdots$.

Let $\psi_n\colon\ \mathcal{E}(M_{p_n}, M_{q_n}) \to \mathcal{D}_\omega$ unital *-morphisms, where $p_n, q_n \geq n$. Then

$$\psi_\omega\colon\ \prod_\omega \mathcal{E}(M_{p_n}, M_{q_n}) \to (\mathcal{D}_\omega)_\omega$$

is a unital morphism. One can see, that there is a unital *-morphism from $\mathcal{Z}$ into $\prod_\omega \mathcal{E}(M_{p_n}, M_{q_n})$. (If $\lim_\omega \gcd(p_n, q_n) = \infty$ this is trivial, because then it contains an ultrapower of matrix algebras.)

Thus, there is a unital morphism from $\mathcal{Z}$ into $(\mathcal{D}_\omega)_\omega$. On the other hand, $(\mathcal{D}_\omega)_\omega$ is the quotient of $\ell_\infty(\ell_\infty(\mathcal{D})) \cong \ell_\infty(\mathcal{D})$ induced by some other character ω_1 on its center $\ell_\infty(\ell_\infty) \cong \ell_\infty$, i.e. $(\mathcal{D}_\omega)_\omega \cong \mathcal{D}_{\omega_1}$. We obtain that $\mathcal{Z} \subset \mathcal{D}_{\omega_1}$ for some free ultrafilter on $\mathbb{N} \cong \mathbb{N} \times \mathbb{N}$. Since $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D} \otimes \cdots$ and $\mathcal{Z}$ is self-absorbing, $\mathcal{D}$ is $\mathcal{Z}$-absorbing.

(2): By Proposition 4.4(4), A is stably $\mathcal{D}$-absorbing if and only if a copy of $\mathcal{D}$ is unitally contained in $F(A)$. If J is a closed ideal of A, then there are unital *-morphisms from $F(A)$ into $F(J)$ and from $F(A)$ onto $F(A/J)$, cf. Remark 1.15(3). Thus, J and A/J are stably $\mathcal{D}$-absorbing if A is $\mathcal{D}$-absorbing.

If $E \subset A$ is a hereditary C^*-subalgebra and if J denotes the closed ideal of A generated by E, then $\mathcal{K} \otimes E \cong \mathcal{K} \otimes J$. Hence, E is stably $\mathcal{D}$-absorbing if A is stably $\mathcal{D}$-absorbing.

Suppose that $A = \operatorname{indlim}(h_n\colon B_n \to B_{n+1})$, where $B_1, B_2, \ldots$ are separable. Let $h_{n,\infty}\colon B_n \to A$ denote the corresponding natural morphisms. Then $A_n := h_{n,\infty}(B_n)$ is an increasing sequence of C^*-subalgebras of A, such that $\bigcup_n A_n$ is dense in A. If B_n is stably $\mathcal{D}$-absorbing, then its quotient A_n is stably $\mathcal{D}$-absorbing.

It follows that $\mathcal{D}$ is unitally contained in $F(A_n)$ for $n \in \mathbb{N}$. Since $\mathcal{D}$ and A are separable, we get that $\mathcal{D}$ is unitally contained in $F(A)$ by Proposition 1.14.

Suppose that B is unital and stably $\mathcal{D}$-absorbing, i.e. there is an isomorphism ψ from $\mathcal{K}\otimes B$ onto $\mathcal{K}\otimes B\otimes\mathcal{D}$ that is approximately unitarily equivalent to $a\mapsto a\otimes 1$ for $a\in\mathcal{K}\otimes B$.

Then there exist a unitary $u\in\mathcal{M}(\mathcal{K}\otimes B\otimes\mathcal{D})$ such that

$$u^*\psi(e_{1,1}\otimes 1_B)u = e_{1,1}\otimes 1_B\otimes 1_{\mathcal{D}}\,.$$

Then there is a unique isomorphism φ from B onto $B\otimes\mathcal{D}$ with

$$u^*\psi(e_{1,1}\otimes b)u = e_{1,1}\otimes\varphi(b)$$

for $b\in B$, and φ is approximately unitarily equivalent to $b\mapsto b\otimes 1$.

(3): Suppose that the commutator group $[\mathcal{U}(\mathcal{D}),\mathcal{U}(\mathcal{D})]$ of $\mathcal{U}(\mathcal{D})$ is contained in $\mathcal{U}_0(\mathcal{D})$. Let $z_1,z_2,\ldots$ a sequence that is dense in $\mathcal{U}(\mathcal{D})$. For $n\in\mathbb{N}$ there are $u_n,v_n\in\mathcal{U}(\mathcal{D}\otimes\mathcal{D})$ with

$$\|((v_nu_n)^*u_nv_n)^*\eta_1(z_k)((v_nu_n)^*u_nv_n)-\eta_2(z_k)\|<1/n\,,$$

and there is a continuous map $w\colon\ [0,1/2]\to\mathcal{U}(\mathcal{D}\otimes\mathcal{D})$ with $w_0=1$ and $w_{1/2}=(v_nu_n)^*u_nv_n$. We define unital completely positive maps $T_n\colon\ \mathcal{D}\to\mathcal{E}(\mathcal{D},\mathcal{D})$ by $T_n(d)_t:=(w_t)^*\eta_1(d)(w_t)$ for $t\in[0,1/2]$ and $T_n(d)_t:=(2t-1)\eta_2(d)+2(1-t)T_n(d)_{1/2}$ for $t\in(1/2,1]$. Then T_n is $2/n$-multiplicative on $\{z_1,\ldots,z_n\}$. Thus, the restriction of the ultrapower T_ω to $\mathcal{D}\subset\mathcal{D}_\omega$ defines a unital *-morphism

$$\Psi\colon\ \mathcal{D}\to\mathcal{E}(\mathcal{D},\mathcal{D})_\omega\,.$$

Let A a separable C^*-algebra and J a closed ideal of A such that J and A/J are stably $\mathcal{D}$-absorbing. Then there exist unital subalgebras $D_0\subset F(J)$ and $D_1\subset F(A/J)$ that are isomorphic to $\mathcal{D}$. Thus $\mathcal{E}(\mathcal{D},\mathcal{D})\cong\mathcal{E}(D_0,D_1)$, and, by Proposition 1.17, there exits a unital *-morphism $h\colon\mathcal{E}(\mathcal{D},\mathcal{D})\to F(A)$. The superposition $h_\omega\circ\Psi$ is a unital *-morphism from $\mathcal{D}$ into $F(A)_\omega$. Since $\mathcal{D}$ is simple and separable, there is a copy of $\mathcal{D}$ unitally contained even in $F(A)$ itself, cf. Proposition 1.14 (with $A_n=A$). Hence, A is stably $\mathcal{D}$-absorbing.

Conversely, suppose that the class of separable stably $\mathcal{D}$-absorbing algebras is closed under extensions. Then $\mathcal{E}(\mathcal{D},\mathcal{D})\cong\mathcal{D}\otimes\mathcal{E}(\mathcal{D},\mathcal{D})$, because $\mathcal{E}(\mathcal{D},\mathcal{D})$ is a unital extension of the $\mathcal{D}$–absorbing algebra $\mathcal{D}\oplus\mathcal{D}$ by $C_0(0,1)\otimes\mathcal{D}$:

$$0\to C_0((0,1),\mathcal{D}\otimes\mathcal{D})\to\mathcal{E}(\mathcal{D},\mathcal{D})\to\mathcal{D}\otimes 1\oplus 1\otimes\mathcal{D}\to 0\,.$$

In particular, there is a unital *-morphism $\psi\colon\ \mathcal{D}\to\mathcal{E}(\mathcal{D},\mathcal{D})$. It is given by a point-norm continuous path of unital *-morphisms $\psi_t\colon\ \mathcal{D}\to\mathcal{D}\otimes\mathcal{D}$ with $\psi_0(\mathcal{D})\subset\mathcal{D}\otimes 1$ and $\psi_1(\mathcal{D})\subset 1\otimes\mathcal{D}$. For $u,v\in\mathcal{U}(\mathcal{D})$ we have that $\psi_0(u^*v^*uv)$ is in $\mathcal{U}_0(\mathcal{D}\otimes\mathcal{D})$ by the path $w_t:=\psi_0(u)^*\psi_t(v)^*\psi_0(u)\psi_t(v)$ with $w_0=\psi_0(u^*v^*uv)$ and $w_1=1\otimes 1$. If ι denotes an isomorphism from $\mathcal{D}\otimes\mathcal{D}$ onto $\mathcal{D}$, then $\iota\circ\psi$

is approximately inner. Since $\iota(\psi_0(u^*v^*uv)) \in \mathcal{U}_0(\mathcal{D})$, and since $\mathcal{U}_0(\mathcal{D})$ is a closed and open normal subgroup of $\mathcal{U}(\mathcal{D})$, it follows $u^*v^*uv \in \mathcal{U}_0(\mathcal{D})$.

(4): If the class of stably $\mathcal{D}$-absorbing separable C^*-algebras is closed under extensions, then $[\mathcal{U}(\mathcal{D}),\mathcal{U}(\mathcal{D})] \subset \mathcal{U}_0(\mathcal{D})$. The latter implies that every stably $\mathcal{D}$-absorbing algebra is $\mathcal{D}$-absorbing. □

We conclude this section with some remarks and questions:
(1) If $\eta_1, \eta_2 \colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ are homotopic then for every separable C^*-algebra A there is a natural isomorphism

$$KK(\mathcal{D}, A \otimes \mathcal{D}) \cong K_0(A \otimes \mathcal{D}) .$$

(Here we do not assume that the UCT is valid for $\mathcal{D}$.)
(2) In particular, $KK(\mathcal{D},\mathcal{D}) \cong K_0(\mathcal{D})$ with ring-structure given by tensor product of projections, and $KK^1(\mathcal{D},\mathcal{D}) \cong K_1(\mathcal{D})$.
(3) Let $\mathcal{D}$ be a self-absorbing algebra.
Are η_1 and η_2 homotopic?
Is $\mathrm{cov}(\mathcal{D}) < \infty$? Is $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{Z}$?
Is $\mathcal{U}(\mathcal{D})/\mathcal{U}_0(\mathcal{D}) \to K_1(\mathcal{D})$ an isomorphism if $\mathcal{D}$ is self-absorbing?
Is always $K_1(\mathcal{D}) = 0$ for self-absorbing unital $\mathcal{D}$?
(4) Does there exist a nuclear C^*-algebra A such that A is stably projectionless, that the flip automorphism of $A \otimes A$ is approximately inner (by unitaries in $\mathcal{M}(A \otimes A)$) and with $K_*(A) = K_*(\mathbb{C})$?

A Elementary Properties of Ultrapowers.

One has to take a more general and flexible approach to ultrapowers to get a tool for our proofs: It is useful for our applications to consider bounded subsets X_n of the closed unit-ball of a Banach spaces B_n (or of $\mathcal{L}(B_n, B_n)$). This is the most general form of bounded metric spaces (with an given uniform bound for the diameters). But the needed selection results are part of elementary set theory (and are rather elementary).

Let $\omega \subset \wp(\mathbb{N})$ a (fixed) free ultrafilter on $\mathbb{N}$. Then $X_1 \times X_2 \times \cdots$ with semi-metric

$$d_\omega((s_1, s_2, \ldots), (t_1, t_2, \ldots)) := \lim_\omega \|s_n - t_n\|$$

defines a metric space that is isometric to the (closed) subset

$$X_\omega := \{\pi_\omega(s_1, s_2, \ldots)\,;\ s_1 \in X_1, s_2 \in X_2, \ldots\}$$

of the Banach space

$$\prod_\omega \{B_1, B_2, \ldots\} := \ell_\infty(B_1, B_2, \ldots)/c_\omega(B_1, B_2, \ldots) .$$

We suppose now that on each X_n there is given a sequence of functions $f_n^{(1)}, f_n^{(2)}, \ldots$ with $f_n^{(k)}\colon X_n \to [0,\infty)$ for $k = 1, 2, \ldots$. Further we suppose that for fixed $k \in \mathbb{N}$ the sequence has a common estimate $\gamma_k < \infty$ for the Lipschitz constants of $f_n^{(k)}$ for $n = 1, 2, \ldots$. (This condition can be relaxed in applications by ω-lim-existence conditions.)

We can define functions $f_\omega^{(k)}\colon X_\omega \to [0,\infty]$ for $k \in \mathbb{N}$ by

$$f_\omega^{(k)}(\pi_\omega(s_1, s_2, \ldots)) := \omega\text{-}\lim_n f_n^{(k)}(s_n)\,,$$

because $\omega\text{-}\lim_n f_n^{(k)}(t_n) - f_n^{(k)}(s_n) = 0$, if $\omega\text{-}\lim_n \|t_n - s_n\| = 0$.

The basic lemma is:

Lemma A.1 *Let $X_1, X_2, \ldots$ any sequence of sets and suppose that for each $n \in \mathbb{N}$ there is given a sequence $f_n^{(1)}, f_n^{(2)}, \ldots$ of functions $f_n^{(k)}\colon X_n \to [0,\infty)$ for $k = 1, 2, \ldots$. For $k \in \mathbb{N}$, let*

$$f_\omega^{(k)}(s_1, s_2, \ldots) := \omega\text{-}\lim_n f_n^{(k)}(s_n)\,.$$

Suppose that for every $m \in \mathbb{N}$ and $\varepsilon > 0$, there is $s = (s_1, s_2, \ldots) \in X_1 \times X_2 \times \cdots$ with $f_\omega^{(k)}(s) < \varepsilon$ for $k = 1, \ldots, m$.

Then there is $t = (t_1, t_2, \ldots) \in X_1 \times X_2 \times \ldots$ with

$$\omega\text{-}\lim_n f_n^{(k)}(t_n) = f_\omega^{(k)}(t) = 0$$

for all $k \in \mathbb{N}$.

Moreover, then there is a sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that there are $s_\ell \in X_{n_\ell}$ with $f_{n_\ell}^{(k)}(s_\ell) < 2^{-\ell}$ for $k \leq \ell$, $\ell = 1, 2, \ldots$.

The second part is almost trivial by the fact that any subsequence of a zero-sequence is a zero-sequence. It does not imply the first part because the infinite set $\{n_1, n_2, \ldots\} \subset \mathbb{N}$ is not necessarily contained in the given free ultrafilter ω on $\mathbb{N}$.

Proof. We define subsets $X_{n,m} \subset X_n$ by $X_{n,0} := X_n$

$$X_{n,m} := \{s \in X_n\,;\ \max(f_n^{(1)}(s), \ldots, f_n^{(m)}(s)) < 1/m\}\,.$$

Then $X_{n,m+1} \subset X_{n,m}$. We let $m(n) := \sup\{m \leq n\,;\ X_{n,m} \neq \emptyset\}$. For every $k \in \mathbb{N}$, the set Y_k of $n \in \mathbb{N}$ with $k < m(n)$ is in the free ultrafilter ω, because there are $s_n \in X_n$ with $\omega\text{–}\lim_n f_n^{(j)}(s_n) < (2k)^{-1}$ for $1 \leq j \leq k+1$. $Y_k \in \omega$ (for all $k \in \mathbb{N}$) implies $\omega\text{–}\lim_n 1/m(n) = 0$.

By definition of $m(n)$ we find $t_n \in X_{n,m(n)} \subset X_n$. Then $\omega\text{–}\lim_n f_n^{(j)}(t_n) = 0$ for every $j \in \mathbb{N}$, because $f_n^{(j)}(t_n) \leq 1/m(n)$ for $n > j$.

Second part: We find $n_1 < n_2 < \ldots$ with $m(n_\ell) > 2^\ell$, because $\omega\text{–}\lim_n 1/m(n) = 0$. Now let $s_\ell := t_{n_\ell} \in X_{n_\ell}$. □

A special case of Lemma A.1 is:

Remark A.2 *Let ω a (fixed) free ultrafilter on $\mathbb{N}$, and let X a bounded subset of a Banach space B. Suppose that $f_1, f_2, \ldots$ is a sequence of functions $f_k : X \to [0,2]$.*

If for every $m \in \mathbb{N}$ and $\varepsilon > 0$ there is a sequence $s_1, s_2, \ldots \in X$ such that $\omega\text{-}\lim_n f_k(s_n) < \varepsilon$ for $k = 1, \ldots, m$, then there is a sequence $(t_1, t_2, \ldots)$ in X such that $\lim_n f_k(t_n) = 0$ for all $k \in \mathbb{N}$.

Let A a C^*-algebra, $0 < \gamma < \infty$ and suppose that $X_n \subset \mathcal{L}(A)$ are subsets with $\|T\| \leq \gamma$ for all $T \in X_n$ $(n = 1, 2, \ldots)$. Then $\prod_\omega \{X_1, X_2, \ldots\}$ denotes the set of ultrapowers $T_\omega \colon A_\omega \to A_\omega$ for $T_\omega = (T_1, T_2, \ldots)_\omega$ defined by $T_\omega(\pi_\omega(a_1, a_2, \ldots)) := \pi_\omega(T_1(a_1), T_2(a_2), \ldots)$ where $(a_1, a_2, \ldots) \in \ell_\infty(A)$ and $T_n \in X_n$ for all $n \in \mathbb{N}$.

Lemma A.3 *Suppose that $C \subset A_\omega$ is a separable subset, $0 < \gamma < \infty$ and $X_n \subset \mathcal{L}(A)$ are subsets with $\|T\| \leq \gamma$ for all $T \in X_n$ and $n = 1, 2, \ldots$.*

Then the set of restricted maps $T_\omega | C \colon C \to A_\omega$ with $T_\omega \in \prod_\omega \{X_1, X_2, \ldots\}$ is point-norm closed.

Proof. Let $S \colon C \to A_\omega$ a map with the property that for every finite sequence $c^{(1)}, \ldots, c^{(m)} \in C$ and $\varepsilon > 0$ there is $T_\omega \in \prod_\omega \{X_1, X_2, \ldots\}$ with

$$\|S(c^{(j)}) - T_\omega(c^{(j)})\| < \varepsilon$$

for $j = 1, \ldots, m$. We get that S has Lipschitz constant $< 2\gamma$.

Let $c^{(1)}, c^{(2)}, \ldots$ a dense sequence in C, and $(a_1^{(j)}, a_2^{(j)}, \ldots) \in \ell_\infty(A)$, $(b_1^{(j)}, b_2^{(j)}, \ldots) \in \ell_\infty(A)$ representing sequences for $c^{(j)}$ respectively $S(c^{(j)})$, $j = 1, 2, \ldots$. Then the functions $f_n^{(j)}(T) := \|b_n^{(j)} - T(a_n^{(j)})\|$ on X_n satisfy the assumptions of Lemma A.1. Thus, there are $S_n \in X_n$ with $S_\omega(c^{(j)}) = S(c^{(j)})$ for all $j \in \mathbb{N}$. Since S_ω and S are Lipschitz, it follows that $S = S_\omega | C$. □

Proposition A.4 *Suppose that B is a C^*-algebra and J a closed ideal of B, that $P_1, P_2, \ldots$ is a sequence of polynomials in in non-commuting variables x, x^* with coefficients in B_ω, that $\mathcal{V}_n \subset \mathcal{L}(B)$ are subsets of linear operators of norm $\leq \gamma < \infty$, and that $C \subset B_\omega$ is a separable subset.*

If for each $n \in \mathbb{N}$, $\varepsilon > 0$ and every finite subset $Y \subset C$, there is a contraction $a \in J_\omega$ with $\|P_k(a, a^*)\| < \varepsilon$ for $k = 1, \ldots, n$, and $\|S_\omega(y) - a^* y a\| < \varepsilon \cdot \|y\|$ for suitable $S_n \in \mathcal{V}_n$ and all $y \in Y$.

Then there exist $T_n \in \mathcal{V}_n$ $(n = 1, 2, \ldots)$ and a contraction $x_0 \in J_\omega$ with $P_k(x_0, x_0^*) = 0$ for all $k \in \mathbb{N}$ and $T_\omega(c) = x_0^* c x_0$ for all $c \in C$.

Suppose that, in addition, $A \subset B_\omega$ is σ-unital (respectively is separable) and $a \in \mathrm{Ann}(A) \cap J_\omega$ (respectively $a \in (A, B)^c \cap J_\omega = A' \cap J_\omega$) then there is $x_0 \in \mathrm{Ann}(A) \cap J_\omega$ (respectively $x_0 \in (A, B)^c \cap J_\omega$) with $P_k(x_0, x_0^*) = 0$ for all $k \in \mathbb{N}$.

If one takes as $\mathcal{V}_n$ the set of maps $b \mapsto d^* b d$ with a contraction $d \in B$ (respectively $J = B$), then the assumption on $\mathcal{V}_n$ and a (respectively on J and a) are trivially satisfied if $\max_{k \leq n} \|P_k(a^*, a)\| < \varepsilon$.

Proof. The linear operators $S_\omega\colon B_\omega \to B_\omega$ for $S_\omega := (S_1, S_2, \ldots)_\omega$ with $S_n \in \mathcal{V}_n$ have norm $< 2\gamma$. Let $c^{(1)}, c^{(2)}, \ldots$ a dense sequence in C. We find representing sequences $c_1^{(k)}, c_2^{(k)}, \ldots \in B$ for $c^{(k)}$ with $\|c_n^{(k)}\| \le \|c^{(k)}\|$, $k, n \in \mathbb{N}$.

$P_k(x^*, x)$ is the sum of products of $d^{(k,j)} \in B_\omega$, x and x^*. $j = 1, \ldots, \ell_k$. There are representing sequences $d_1^{(k,j)}, d_2^{(k,j)}, \ldots \in B$ of $d^{(k,j)}$ with norms $\le \|d^{(k,j)}\|$. The corresponding non-commutative polynomials $P_n^{(k)}(x^*, x)$ with coefficients in B have the property that $\sup_n \|P_n^{(k)}(b_n^*, b_n)\| < \infty$ for every $(b_1, b_2, \ldots) \in \ell_\infty(B)$ and satisfy

$$\pi_\omega(P_1^{(k)}(b_1^*, b_1), P_2^{(k)}(b_2^*, b_2), \ldots) = P_k(b_\omega^*, b_\omega)\,.$$

Let $X_n = \mathcal{V}_n \times \{b \in J\,;\ \|b\| \le 1\}$ for $n \in \mathbb{N}$. We define

$$f_n^{(k)}(T, b) := \|T(c_n^{(k)}) - b^* c_n^{(k)} b\| + \|P_n^{(k)}(b^*, b)\|$$

for $(T, b) \in X_n$ and $k = 1, 2, \ldots$.

Then $(X_n, f_n^{(1)}, f_n^{(2)}, \ldots)$ $(n = 1, 2, \ldots)$ satisfies the assumptions of Lemma A.1.

Thus there exists $t = ((T_1, b_1), (T_2, b_2), \ldots) \in X_1 \times X_2 \times \cdots$ with $\omega\text{-}\lim_n f_n^{(k)}(T_n, b_n) = 0$. Then $T_\omega = (T_1, T_2, \ldots)_\omega$ and $x_0 := \pi_\omega(b_1, b_2, \ldots)$ are as desired.

To get x_0 in $\mathrm{Ann}(A, B) \cap J_\omega$ or in $(A, B)^c$ we have to add to the polynomials $P_1, P_2, \ldots$ the polynomials $Q_1(x, x^*) = x a_0$ and $Q_2(x, x^*) = a_0 x$ respectively $Q_n(x, x^*) = x a_n - a_n x$, where $a_0 \in A$ is a strictly positive contraction and $a_1, a_2, \ldots$ is dense in the unit ball of A. □

Lemma A.5 *If $T_1, T_2, \ldots \in \mathcal{L}(B, B)$ is a bounded sequence of positive maps and $A \subset B_\omega$ is a σ-unital C^*-subalgebra. Then there are contractions $b_1, b_2, \ldots \in B_+$ such that $\|S_n\| \le \|T_\omega | A\|$ and $S_\omega | A = T_\omega | A$ for $S_n := T_n(b_n(.)b_n)$.*

Proof. Let $d \in A_+$ a strictly positive contraction for A and let $e = (e_1, e_2, \ldots) \in \ell_\infty(B)$ a positive contraction with $\pi_\omega(e) = d$. Then $\|T_\omega(d^{1/k})\| \le \|T_\omega | A\| =: \gamma$ for all $k \in \mathbb{N}$.

Let $X_n := \{t e_n^{1/j}\,;\ j \in \mathbb{N},\ 0 < t \le 1\}$ and consider the functions $f_n^{(k)}(b) := \max(\|e_n^{1/k} - b e_n^{1/k}\|, \|T_n(b^2)\| - \gamma)$ on $X_n \subset B$.

Then $(X_n, f_n^{(1)}, f_n^{(2)}, \ldots)$ $(n = 1, 2, \ldots)$ satisfy the assumptions Lemma A.1, because $\|e^{1/j} e^{1/k} - e^{1/k}\| \le k/j$ and $\|T_\omega(e^{2/j})\| \le \gamma$ for $j \in \mathbb{N}$.

By Lemma A.1, there is a positive contraction $g = (g_1, g_2, \ldots) \in \ell_\infty(A)$ with $g_n \in X_n$ such that $\pi_\omega(g) d = d$ and $\|T_n(g_n^2)\| \le 1$. Thus, $S_\omega | D = T_\omega | D$ for $D := \overline{d A_\omega d} \supset A$ and $S_n := T_n(g_n(\cdot) g_n)$. □

B Proofs of Results in Section 1

Proof (of Proposition 1.3).

Let $a_1, a_2, \ldots \in A_+$ a sequence that is dense in the set of positive contractions in A.

Consider the non-commutative polynomials $P_1(x, x^*) := x^* - x$, $P_2(x, x^*) := a - x^*xa$, $P_3(x, x^*) := (b+c)x^*x$, $P_{3+n}(x, x^*) := a_n x - x a_n$ for $n = 1, 2, \ldots$. An approximate zero for the polynomials $P_k(x, x^*)$ is given by $x_n = a^{1/n}$: $P_k(x_n, x_n^*) = 0$ for $k \neq 2$ and $\|P_2(x_n, x_n^*)\| \leq 2/n$. Thus, by Proposition A.4, there is a self-adjoint contraction $e' \in A' \cap B_\omega$ with $a = e'e'a$ and $(b+c)e' = 0$. Thus $e := e'e' \in (A, B)^c$ is a positive contraction with $ea = a$ and $eb = ec = 0$.

If $z \in A_+$ is a strictly positive element of A, then almost the same argument shows that there is a positive contraction $p \in B_\omega$ with $p(z + b) = z + b$, i.e. $py = yp = y$ for all $y \in C^*(A, b)$.

Let I a closed ideal of B with $b \in I_\omega$, and let $S_1, S_2, \ldots \in \mathcal{V}$ with $S_\omega(c) = bcb$. Consider the non-commutative polynomials $Q_1 := P_1$, $Q_2(x, x^*) := b - x^*xb$, $Q_3(x, x^*) = (e + c)x^*x$ $Q_{3+n} := P_{3+n}$ for $n = 1, 2, \ldots$.

We show below that the sequence of polynomials $(Q_1, Q_2, \ldots)$ have contractive approximate solutions $x_n \in I_\omega$ such that for every $n \in \mathbb{N}$ there is a sequence $S_1^{(n)}, S_2^{(n)}, \ldots$ of contractions in $\mathcal{V}$ with $x_n^* y x_n = S_\omega^{(n)}(y)$ for all $y \in A$.

By Proposition A.4, there exist contractions $T_n \in \mathcal{V}$ $(n = 1, 2, \ldots)$ and a contraction $f' \in I_\omega$ with $P_k(f', (f')^*) = 0$ for all $k \in \mathbb{N}$ and $T_\omega(c) = (f')^* c f'$ for all $c \in A$. Thus $(f')^* = f' \in (A, B)^c$, and $f := f'f' \in (A, B)^c$ is a positive contraction in $A' \cap I_\omega$ with $fe = fc = 0 = b - fb$ and $T_\omega(c) = cf$ for all $c \in A$. In particular, $fa = fea = 0$.

Let $E := C^*(A, b)$, $K := \overline{\mathrm{span}(EbE)}$. Then K is a closed ideal of E, $E = A + K$, $K \subset I_\omega$ and K is the closed span of $\bigcup_n (bA + bAb + Ab)^n$. It follows that every element $d \in K$ is the limit of finite sums $d_n = \sum_n u_n b v_n$ with $u_n \in A \cup \{p\}$ and $v_n \in E \subset B_\omega$. Furthermore, $bEe = \{0\} = bEc$, because $b(A + \mathbb{C}b)^n e = \{0\}$ and $b(A + \mathbb{C}b)^n c = \{0\}$ for $n \in \mathbb{N}$. Thus $(e + c)K = K(e+c) = \{0\}$. Since E is separable, K contains a strictly positive contraction $h \in K_+$.

We find in $C^*(h)_+ \subset K_+$ a sequence of positive contractions $x_1, x_2, \ldots$ with $x_n x_{n+1} = x_n$, $\|h - x_n h\| < 1/n$ and $\lim_{n\to\infty} \|x_n c - c x_n\| = 0$ for all $c \in E$, cf. [29, thm. 3.12.14]. Note that $x_n(e + h) = 0$ for all $n \in \mathbb{N}$, that $\lim \|b - x_n^* x_n b\| = 0$ and $x_n \in I_\omega$.

We show that for every $d \in K$ there is a sequence $R_1, R_2, \ldots \in \mathcal{V}$ with $\sup_n \|R_n\| \leq \|d\|^2$ and $R_\omega(y) = d^* y d$: By assumption there is a bounded sequence $S_1, S_2, \ldots \in \mathcal{V}$ with $S_\omega(y) = b^* y b$ for all $y \in A$. Let (first) d be a finite sum $d = \sum_n u_n b v_n$ with $u_n \in A \cup \{p\}$ and $v_n \in E \subset B_\omega$, and let $(u_1^{(n)}, u_2^{(n)}, \ldots)$ and $(v_1^{(n)}, v_2^{(n)}, \ldots)$ in $\ell_\infty(B)$ be representing sequences for u_n respectively v_n, with $\|v_k^{(n)}\| \leq \|v_n\|$ and $\|u_k^{(n)}\| \leq \|u_n\|$. Then the map R_k, defined by

$$R_k(y) := \sum_{m,n} (v_k^{(m)})^* S_k((u_k^{(m)})^* y u_k^{(n)}) v_k^{(n)} ,$$

is in $\mathcal{V}$, $\|R_k\| \leq \|S_k\| (\sum_n \|v_n\|)^2 (\sum_n \|u_n\|)^2$ and

$$R_\omega(y) = \sum_{m,n} (v_m)^* S_\omega((u_m)^* y u_n) v_n .$$

Since $py = yp = y$ for $y \in A$, we get $R_\omega(y) = d^*yd$ for $y \in A$.

By Lemma A.5 we find another sequence $R'_1, R'_2, \ldots \in \mathcal{V}$ with $R'_\omega(y) = d^*yd$ for $y \in A$ and $\|R'_n\| \leq \|d\|^2$. This happens for every $d \in K$ by Lemma A.3, because every $d \in K$ can be approximated in norm by finite sums $\sum_n u_n b v_n$ of the above considered type. Thus, Proposition A.4 applies to $Q_1, Q_2, \ldots, \mathcal{V}_n := \mathcal{V}$, $\gamma = 1$ and I (in place of J there).

Now we can repeat the above arguments with $c, J, e + f, CB(B, B)$ and $c - x^*xc, (f + e)x^*x$ in place of $b, I, e + c, \mathcal{V}$ and Q_2, Q_3 . We get a self-adjoint contraction $g' \in A' \cap J_\omega$, such that with $g := g'g' \in (A, B)^c$, $g \in J_\omega$, $gc = c$, $ge = gf = 0$. Then e, f, g are as stipulated. □

Proof (of Proposition 1.6). Suppose that A is a separable C^*-subalgebra of C. The set of all positive elements in $A' \cap I$ of norm < 1 build an an approximate unit for I by Definition 1.5.

Let $b \in C_+$ with $\pi_I(b) \in \pi_I(A)' \cap C/I$. Then $ab - ba \in I$ for all $a \in A$. $[b, A]$ is contained in a separable C^*-subalgebra D of I. Let $d \in D_+$ strictly positive. Since I is a σ-ideal of C there exists a positive contraction $e \in A' \cap I$ with $ed = d$. Then $c := (1 - e)b(1 - e)$ satisfies $c \in A' \cap C$ and $\pi_I(c) = \pi_I(b)$. Thus

$$0 \to A' \cap I \to A' \cap C \to \pi_I(A)' \cap (C/I) \to 0$$

is short exact.

Let $D \subset \pi_I(A)' \cap (C/I)$ a separable C^*-subalgebra and $B \subset A' \cap C$ a separable C^*-algebra with $\pi_I(B) = D$. If d denotes a strictly positive element of $B \cap I$, then there is a positive contraction $e \in C^*(A \cup B)' \cap I$ with $ed = d$.

There is a *-morphism $\lambda\colon\ C_0(0,1] \otimes B \to A' \cap C$ with $\lambda(f_0^n \otimes b) = (1 - e)^n b$ for $b \in B$ and $n \in \mathbb{N}$. It follows $\lambda(C_0(0,1] \otimes (B \cap I)) = \{0\}$ and $\pi_I(\lambda(f)) = \pi_I(f(1))$ for $f \in C_0((0,1], B) \cong C_0(0,1] \otimes B$. Thus there is a *-morphism $\psi\colon\ C_0((0,1] \otimes D) \to A' \cap C$ with $\psi(f_0 \otimes h) = \lambda(f_0 \otimes b)$ for $b \in B$ with $\pi_I(b) = h$. ψ satisfies $\pi_I \circ \psi(f_0 \otimes h) = h$ for $h \in D$.

Since $\mathrm{Ann}(\pi_I(A), C/I) \subset \pi_I(A)' \cap (C/I)$, for every positive $f \in \mathrm{Ann}(\pi_I(A), C/I)$ there is a positive element $b \in A' \cap C$ with $\pi_I(b) = f$. Let $a_0 \in A_+$ a strictly positive element of A. Then $ba_0 \in I$. There is a positive contraction $e \in C^*(b)' \cap I$ with $eba_0 = ba_0$. It follows that $c := b(1 - e) \in C_+$ satisfies $ca_0 = 0$ and $\pi_I(c) = f$. □

Lemma B.1 *Suppose that A is a σ-unital non-degenerate C^*-subalgebra of a C^*-algebra D, that $E \subset A$ is a full and hereditary σ-unital C^*-subalgebra of A, and let $D_E := \overline{EDE}$. Then the natural map from $A' \cap \mathcal{M}(D)$ into $E' \cap \mathcal{M}(D_E)$ is a *-isomorphism (onto $E' \cap \mathcal{M}(D_E)$).*

Proof. The natural *-morphism is given by $\iota(T)c = Tc$ for $T \in A' \cap \mathcal{M}(D)$ and $c \in D_E$. $TD_E \subset D_E$, because T commutes with $E \subset A$. If $\iota(T) = 0$, then $TAEA = ATEA = \{0\}$ because T commutes with A. It follows $TA = \{0\}$ and $T = 0$, because $\mathrm{span}(AEA)$ is dense in A and $\mathrm{span}(AD)$ is dense in D. Thus ι is a *-monomorphism from $A' \cap \mathcal{M}(D)$ into $E' \cap \mathcal{M}(D_E)$, and it suffices to construct a *-morphism $\kappa\colon\ E' \cap \mathcal{M}(D_E) \to A' \cap \mathcal{M}(D)$ with $\iota \circ \kappa = \mathrm{id}$.

One can see, that $(A\otimes\mathcal{K})'\cap\mathcal{M}(D\otimes\mathcal{K}) = (A'\cap\mathcal{M}(D))\otimes 1$ and $A'\cap\mathcal{M}(D) = \mathcal{M}(A)'\cap\mathcal{M}(D)$ for all non-degenerate pairs $A\subset D$.

There is an element $g\in A\otimes\mathcal{K}$ such that g^*g is a strictly positive element of $A\otimes\mathcal{K}$ and gg^* is a strictly positive element of $E\otimes\mathcal{K}$, cf. [7]. The polar decomposition $g = v(g^*g)^{1/2} = (gg^*)^{1/2}v$ of g in $(A\otimes\mathcal{K})^{**}$ defines an isomorphism ψ from $E\otimes\mathcal{K}$ onto $A\otimes\mathcal{K}$ by $\psi(e) := v^*ev$. Clearly, ψ extends to an isomorphism from $\mathcal{M}(D_E\otimes\mathcal{K})$ onto $\mathcal{M}(D\otimes\mathcal{K})$ such that $\psi(T) = v^*Tv$ in $(D\otimes\mathcal{K})^{**}$. It maps to $\mathcal{M}(D\otimes\mathcal{K})$ because $\psi(T)x = \lim_n(g^*g+1/n)^{-1/2}g^*Tg(g^*g+1/n)^{-1/2}x$ for $x\in D\otimes\mathcal{K}$.

If $T\in\mathcal{M}(D_E\otimes\mathcal{K})$ commutes with $E\otimes\mathcal{K}$, then $\psi(T)$ commutes with $A\otimes\mathcal{K}$ and $\psi(T)y = Ty$ for all $y\in D_E\otimes\mathcal{K}$, because $Tg(g^*g+1/n)^{-1/2}(gg^*)^{1/k}y = g(g^*g+1/n)^{-1/2}(gg^*)^{1/k}Ty$ for all $y\in D_E\otimes\mathcal{K}$.

Thus, there is a *-morphism κ from $E'\cap\mathcal{M}(D_E)$ into $A'\cap\mathcal{M}(D)$ with $\kappa(S)\otimes 1 = \psi(S\otimes 1)$. We have $\iota(\kappa(S))(c)\otimes p = \psi(S\otimes 1)(c\otimes p) = (S\otimes 1)(c\otimes p)$ for $c\in D_E$ and $p\in\mathcal{K}$. Hence $\iota\circ\kappa = \mathrm{id}$. □

Proof (of Proposition 1.9). (1) is obvious.

(2)+(3): Let $Y = \{y_1, y_2, \ldots\}\subset B_\omega$, $a_0\in A_+$ a strictly positive element of A, and $c := (1+\|d\|)^{-1}d$ with $d := a_0 + \sum_n 2^{-n}(1+\|y_n\|^2)^{-1}(y_ny_n^* + y_n^*y_n)$. By Corollary 1.7 there exists a positive contraction $e\in B_\omega$ with $ec = c$. Thus $ea_0 = a_0 = a_0e$ and $ey = y = ye$ for all $y\in Y$.

If $e\in B_\omega$ is any positive contraction with $ea_0 = a_0$ then $ea = a = ae$ for all $a\in D_{A,B}\supset A$. In particular, $e\in(A,B)^c$. If $b\in(A,B)^c\subset\{a_0\}'\cap B_\omega$, then $(eb-b)$ and $(be-b)$ are in $\mathrm{Ann}(a_0, B_\omega) = \mathrm{Ann}(A, B_\omega)$. Thus $e+\mathrm{Ann}(A,B_\omega) = 1$ in $F(A,B)$.

(4): The natural *-morphism is given by

$$b\in\mathcal{N}(D_{A,B})\mapsto L_b\in\mathcal{M}(D_{A,B})\subset\mathcal{L}(D_{A,B})\,,$$

where $L_b(a) := ba$ for $a\in D_{A,B}$, and involution on $\mathcal{M}(D_{A,B})$ is defined by $t^*(a) := t(a^*)^*$ for $a\in D_{A,B}$ and $t\in\mathcal{M}(D_{A,B})$. Clearly, this is a *-morphism with kernel $\mathrm{Ann}(A,B_\omega)$. $D_{A,B}$ embeds naturally into $\mathcal{M}(D_{A,B})$ by $b\mapsto L_b$ for $b\in D_{A,B}$.

Let $t\in\mathcal{M}(D_{A,B})_+$ and let $a_0\in A$ a strictly positive contraction. Then $c_n := a_0^{1/n}ta_0^{1/n}\in D_{A,B}$ converges to t in the strict topology. In particular, $L_{c_n}\colon C^*(a_0)\to B_\omega$ converges in point-norm topology to $t|C^*(a_0)$.

Let $\mathcal{S}$ denote the set of maps $L_b\colon B\to B$ with $b\in B_+$ and $\|b\|\le 1$. Then $L_{c_n}\in\mathcal{S}^\omega$ for every $n\in\mathbb{N}$. By Lemma A.3 (or by [22, proof of lem. 2.13]) there exists a sequence $L_{b_n}\in\mathcal{S}$ with $t|C^*(a_0) = (L_{b_1}, L_{b_2},\ldots)_\omega|C^*(a_0)$. Thus, $b := \pi_\omega(b_1, b_2,\ldots)\in B_\omega$ satisfies $b\ge 0$ and $ba_0^{1/n} = t(a_0^{1/n})$ for $n\in\mathbb{N}$. Since, a_0 is a strictly positive element of $D_{A,B}$, it follows that $b\in\mathcal{N}(D_{A,B})$ and $L_b = t$.

(5): Since $\mathrm{Ann}(A,B_\omega) = \mathrm{Ann}(D_{A,B},B_\omega)$, the kernel is $\mathrm{Ann}(A,B_\omega)\subset(A,B)^c$. Clearly, the image of $(A,B)^c$ in $\mathcal{M}(D_{A,B})$ commutes with A. If $c\in\mathcal{M}(D_{A,B})$ commutes with A and is the image of $b\in\mathcal{N}(D_{A,B})$, then $[b,A]\subset$

$\mathrm{Ann}(A, B_\omega)$. Thus $[b, a_1a_2] = 0$ for $a_1, a_2 \in A$. Since $A = A \cdot A$, it follows $b \in (A, B)^c$. Hence the natural epimorphism from $\mathcal{N}(D_{A,B})$ onto $\mathcal{M}(D_{A,B})$ defines a *-isomorphism η from $F(A, B) = (A, B)^c/\mathrm{Ann}(A, B_\omega)$ onto $A' \cap \mathcal{M}(D_{A,B})$ with $\rho_{A,B}(g \otimes a) = \eta(g)a$ for $g \in F(A, B)$ and $a \in A$.

(6): If $e = e^2 \geq 0$ is the unit of $(A, B)^c$, and $b \in B_\omega$ is a positive contraction with $be = 0$, then $e + \mathrm{Ann}(A, B_\omega)$ is the unit of $F(A, B)$ and $ba = b\rho_A(1 \otimes a) = bea = 0$ for $a \in A$, i.e. $b \in \mathrm{Ann}(A, B_\omega)$. Since $\mathrm{Ann}(A, B_\omega)$ a closed ideal of $(A, B)^c$, it follows $b = 0$. Thus e is the unit of B_ω.

If f is the unit element of B_ω and $(f_1, f_2, \ldots) \in \ell_\infty(B)$ is a representing sequence of positive contractions for f, then $g := \sum_n 2^{-n} f_n$ satisfies $\|g\| \leq 1$ and $f_n \leq 2^{1/n} g^{1/n^2}$. Hence $f = h$ for $h := \pi_\omega(g^{1/4}, g^{1/9}, \ldots)$. It follows that zero can not be in the spectrum of g, i.e. that B is unital.

The other implications are obvious.

(7): Clearly, if B is unital and $1_B \in A$, then $\mathrm{Ann}(A, B_\omega) = \{0\}$.

If $\mathrm{Ann}(A, B_\omega) = \{0\}$ then $(A, B)^c \cong F(A, B)$. Thus $(A, B)^c$ and B are unital by parts (1) and (6). Let $a_0 \in A$ is a strictly positive contraction for A, then $1_B \in D_{A,B} = \overline{a_0 B_\omega a_0}$ by Remark 2.7. It follows that a_0 is invertible in B_ω, i.e. $1_B \in A$.

(8): Let $E := \overline{dAd}$. Then E is a full σ-unital hereditary C^*-subalgebra of A and $\overline{dD_{A,B}d} = D_{E,B} = \overline{ED_{A,B}E}$.

A natural *-morphism ι from $A' \cap \mathcal{M}(D_{A,B})$ into $E' \cap \mathcal{M}(D_{E,B})$ is given by $\iota(T)c := Tc$ for $T \in A' \cap \mathcal{M}(D_{A,B})$ and $c \in D_{E,B}$. It is a *-isomorphism from $A' \cap \mathcal{M}(D_{A,B})$ onto $E' \cap \mathcal{M}(D_{E,B})$ by Lemma B.1, because A is a σ-unital non-degenerate subalgebra of $D_{A,B}$, $E \subset A$ is a full hereditary σ-unital C^*-subalgebra of A, and $D_{E,B} = \overline{ED_{A,B}E}$.

Let $\eta_1\colon F(A, B) \to A' \cap \mathcal{M}(D_{A,B})$ and $\eta_2\colon F(E, B) \to E' \cap \mathcal{M}(D_{E,B})$ the isomorphisms from part (5), then $\psi := \eta_2^{-1} \circ \iota \circ \eta_1$ is a *-isomorphism from $F(A, B)$ onto $F(E, B)$ with $\rho_{E,B}(\psi(g) \otimes a) = \rho_{A,B}(g \otimes a)$ for $a \in E \subset A$ and $g \in F(A, B)$.

(9): Suppose that $C \subset B$ is a hereditary C^*-subalgebra with $A \subset C_\omega \subset B_\omega$. Then $D_{A,C} = D_{A,B} \subset C_\omega$. Since A is σ–unital, the natural *-morphisms $\mathcal{N}(D_{A,C}) \to \mathcal{M}(D_{A,C})$ and $\mathcal{N}(D_{A,B}) \to \mathcal{M}(D_{A,C})$ are epimorphisms by part (4), and map $(A, C)^c$ respectively $(A, B)^c$ onto $A' \cap \mathcal{M}(D_{A,C})$. Thus $(A, B)^c = (A, C)^c + \mathrm{Ann}(A, B_\omega)$. Because $\mathrm{Ann}(A, C_\omega) = \mathrm{Ann}(A, B_\omega) \cap C_\omega$, it follows $F(A, B) \cong F(A, C)$. □

Proof (of Proposition 1.12). Let H_∞ denote the free semi-group on countably many generators $X := \{x_1, x_2, \ldots\}$ with involution given by $(y_1 \cdot y_2 \cdots y_n)^* := y_n \cdots y_2 \cdot y_1$ for $y_i \in X$, and let $C^*(H_\infty)$ be the full C^*-hull $C^*(\ell_1(H_\infty))$ of the Banach *-algebra $\ell_1(H_\infty)$. $C^*(H_\infty)$ is projective in the category of all C^*-algebras.

Since $(C^*(A, B), B)^c \subset (A, B)^c$ and $\mathrm{Ann}(C^*(A, B), B_\omega) \subset \mathrm{Ann}(A, B_\omega)$, it suffices to consider the case where $B \subset A$ to get (1) also for general separable $A \subset B_\omega$. So we proof the strong result (2) in case $B \subset A$.

Let $a_0 \in A_+$ a strictly positive contraction for A with $\|a_0\| = 1$. $b_1 := a_0, b_2, \ldots \in A_+$, $d_1 := 1, d_2, \ldots \in D_+$ sequences that are dense in the set of positive contractions of norm one in A respectively in D, and let $f_0 \in B_+$ denote a strictly positive contraction for B. For each $n \in \mathbb{N}$ there are

(1) a sequence $c_1^{(n)}, c_2^{(n)}, \ldots \in B_+$ with $\pi_\omega(c_1^{(n)}, c_2^{(n)}, \ldots) = b_n$ and $\|c_k^{(n)}\| = 1$,
(2) a sequence $e_1^{(n)}, e_2^{(n)}, \ldots \in B_+$ with $e_n := \pi_\omega(e_1^{(n)}, e_2^{(n)}, \ldots) \in B^c$, $\|e_k^{(n)}\| = 1$, and $e_n + \mathrm{Ann}(B) = d_n$, and
(3) a sequence $\mu_1^{(n)}, \mu_2^{(n)}, \ldots \in B^*$ of pure states on B with $\mu_\omega^{(n)}(f_0 e_n) = \|f_0 e_n\| = \|\rho_B(d_n \otimes f_0)\|$.

For $k \in \mathbb{N}$, we define *-morphisms $\theta_k \colon C^*(H_\infty) \to B$ by $\theta_k(x_n) := e_k^{(n)}$ for the generators $\{x_1, x_2, \ldots\}$ of H_∞. Further let $G := C^*(e_1, e_2, \ldots)$, and $Y_n := \{c_k^{(j)}\,;\ k, j \leq n\}$.

$$\theta_\omega = (\theta_1, \theta_2, \ldots)_\omega \colon\ h \in C^*(H_\infty) \mapsto \pi_\omega(\theta_1(h), \theta_2(h), \ldots) \in G \subset B^c \subset B_\omega$$

is an epimorphism from $C^*(H_\infty)$ onto G.

Let h_1 a strictly positive contraction for $(\theta_\omega)^{-1}(G \cap \mathrm{Ann}(B))$ and h_2 a strictly positive contraction for $C^*(H_\infty)$ with $\theta_\omega(h_2) + \mathrm{Ann}(B) = 1$ in $F(B)$.

Below we select sub-sequences $(\theta_{k_m})_{m\in\mathbb{N}}$ and $(\mu_{k_m}^{(n)})_{m\in\mathbb{N}}$ of $(\theta_k)_{k\in\mathbb{N}}$ respectively $(\mu_k^{(n)})_{k\in\mathbb{N}}$ (for $n = 1, 2, \ldots$) such that the morphism $\varphi := (\theta_{k_1}, \theta_{k_2}, \ldots)_\omega$ from $C^*(H_\infty) \subset C^*(H_\infty)_\omega$ into $(A, B)^c = A' \cap B_\omega$ satisfies $\varphi(h_1)a_0 = 0$, $\varphi(h_2)a_0 = a_0$ and $\lambda(\varphi(x_n)f_0) = \|\rho_B(d_n \otimes f_0)\|$ for $\lambda := (\mu_{k_1}, \mu_{k_2}, \ldots)_\omega$, i.e.

$$\lim_{m\to\omega} \mu_{k_m}^{(n)}(\theta_{k_m}(x_n)f_0) = \|\rho_B(d_n \otimes f_0)\|\,.$$

Indeed, we define for each $m \in \mathbb{N}$ the subsets $Q_m, R_m, S_m, T_m \subset \mathbb{N}$ as the set of $k \in \mathbb{N}$ with $\|\theta_k(x_j)y - y\theta_k(x_j)\| < 1/m$ for all $y \in Y_m$ and $j \leq m$, $|\mu_k^{(j)}(\theta_k(x_j)f_0) - \|e_j f_0\|| < 1/m$ for $j \leq m$, $\|\theta_k(h_1)b_j^{(1)}\| < 1/m$ for $j \leq m$, respectively $\|b_j^{(1)} - \theta_k(h_2)b_j^{(1)}\| < 1/m$ for $j \leq m$. Then $Q_m, R_m, S_m, T_m \in \omega$ and, hence, $W_m := Q_m \cap R_m \cap S_m \cap T_m \in \omega$. In particular, W_m is infinite. Since $W_1 \supset W_2 \supset W_3 \supset \cdots$ and W_m is *not* finite, we find $k_m \in W_m$ such that $k_1 < k_2 < \cdots$. The sub-sequence $k_1, k_2, \ldots$ is as desired.

The above defined map $h \in C^*(H_\infty) \mapsto \varphi(h) + \mathrm{Ann}(A, B) \in F(A, B)$ maps h_2 to the unit of $F(B)$ and h_1 to zero. Thus it defines a unital *-morphism $\gamma_1 \colon D \to F(A, B)$ with

$$\gamma_1(\theta_\omega(h) + \mathrm{Ann}(B)) = \varphi(h) + \mathrm{Ann}(A, B)$$

for $h \in C^*(H_\infty)$. Since $\mathrm{Ann}(B)$ is an ideal of $B^c \supset (A, B)^c$ and contains $\mathrm{Ann}(A, B_\omega)$ we can compose γ_1 with the morphism $F(A, B) \to F(B)$ and get $\gamma_2 \colon D \to F(B)$ with

$$\gamma_2(\theta_\omega(h) + \mathrm{Ann}(B)) = \varphi(h) + \mathrm{Ann}(B)\,.$$

Then $\|\rho_B(\gamma_2(d_n)\otimes f_0)\| = \|\varphi(x_n)f_0\| \geq \|\rho_B(d_n\otimes f_0)\|$ for $n = 1, 2, \ldots$. Thus, $\|\rho_B(\gamma_2(d)\otimes f_0\| \geq \|\rho_B(d\otimes f_0)\| > 0$ for all $d \in D_+ \setminus \{0\}$, i.e. $\gamma_2\colon D \to F(B)$ is faithful.

By Corollary 1.8 there exists a *-morphism $\psi\colon C_0((0,1], D) \to (A,B)^c$ with $\psi(f) + \mathrm{Ann}(A, B_\infty) = \gamma_2(f(1))$ for $f \in C_0((0,1], D)$. ψ is as desired. □

Proof (of Corollary 1.13). If A is separable and $C, B_1, B_2, \ldots \subset F(A)$ are separable unital C^*-subalgebras, then we get by induction unital separable C^*-subalgebras $C_1 := C \subset C_2 \subset \ldots \subset F(A)$ and unital *-morphisms $\psi_n\colon C_n \otimes^{\max} B_n \to F(A)$ with $\psi_n|1\otimes B_n$ faithful and $\psi_n|C_n = \mathrm{id}$. Here we let $C_{n+1} := \psi_n(C_n \otimes^{\max} B_n)$. This follows from Corollary 1.8 and part (2) of Proposition 1.12 (with B,A and D replaced by A, $C^*(\lambda(C^*((0,1], C_n)), A)$ and D_n respectively).

Note that $C \subset C_2 \subset C_3 \subset \ldots$ and that there is a natural unital *-homomorphism ψ from $C\otimes^{\max}B_1\otimes^{\max}B_2\otimes^{\max}\ldots$ onto the closure of $\bigcup_n C_n \subset F(A)$ with the properties as stipulated. □

Proof (of Proposition 1.14). Let $C^*(H_\infty)$ as in the proof of Proposition 1.12, and let $a_k \in A_k$ a strictly positive contraction of A_k with $\|(1-a_k)a_{k-1}\| < 2^{-k-1}$, and $a_0 := \sum_{k\in\mathbb{N}} 2^{-k}a_k \in A_+$. There are *-morphisms $\varphi_k\colon C^*(H_\infty) \to A_k^c \subset (A_k, A)^c$ such that the morphisms $\psi_k(h) := \varphi(h) + \mathrm{Ann}(A_k) \in F(A_k)$ have the property that

$$\psi_\omega\colon C^*(H_\infty) \to \prod_\omega \{F(A_1), F(A_2), \ldots\}$$

maps $C^*(H_\infty)$ onto A. Let $h_1 \in C^*(H_\infty)_+$ a strictly positive element of the kernel of ψ_ω, and let $h_2 \in C^*(H_\infty)_+$ a strictly positive contraction for $C^*(H_\infty)$ with $\psi_\omega(h_2) = 1$.

Since $C^*(H_\infty)$ is projective, there are *-morphisms $\varphi_n^{(k)}\colon C^*(H_\infty) \to A$ with $(\varphi_1^{(k)}, \varphi_2^{(k)}, \ldots)_\omega = \varphi_k$. It turns out that for suitable $\lambda_m = \varphi_{\ell_m}^{(k_m)}$ holds:

$$\lambda := (\lambda_1, \lambda_2, \ldots)_\omega\colon C^*(H_\omega) \to A_\omega$$

has the properties $\lambda(C^*(H_\omega)) \subset A^c$, $\lambda(h_1)a_0 = 0$ and $\lambda(h_2)a_0 = a_0$. Indeed: apply Remark A.2 with $X = \{\varphi_n^k\,;\ n,k \in \mathbb{N}\} \subset \mathcal{L}(C^*(H_\omega), A)$ and functions $f_k\colon X \to [0,2]$ given by

$$f^k(\varphi) := \max\{\|\varphi(h_1)a_0\|, \|\varphi(h_2)a_0 - a_0\|, \|[\varphi(x_j), b_i]\|\,;\ i,j \leq k\},$$

where $b_1, b_2, \ldots$ is a dense sequence in the unit ball of A. □

Proof (of Corollary 1.16). Clearly, J_ω is an essential ideal of B_ω if J is an essential ideal of B. Since $(A,J)^c = J_\omega \cap (A,B)^c$ is a σ-ideal of $(A,B)^c$ (cf. Corollary 1.7), we get from Proposition 1.6 that $(A,J)^c$ is a non-degenerate C^*-subalgebra of J_ω. If the image $d + \mathrm{Ann}(A, B_\omega)$ in $F(A,B)$ of $d \in (A,B)_+^c$ is orthogonal to $F(A,J)$, then $(A,J)^c d \subset \mathrm{Ann}(A, B_\omega)$.

Let $a_0 \in A_+$ a strictly positive element of A. We have $J_\omega d a_0 = \{0\}$, because $(A,J)^c$ is non-degenerate. Thus $da_0 = 0$ and $d \in \mathrm{Ann}(A, B_\omega)$. Hence $F(A,J)$ is an essential ideal of $F(A,B)$. □

Proof (of Proposition 1.17). Note that $\mathcal{E}(D_0, D_1)$ is naturally isomorphic to the quotient of $\mathrm{cone}(D_0) \otimes^{max} \mathrm{cone}(D_1)$ by the ideal generated by

$$((f_0 \otimes 1_{D_0}) \otimes 1) + (1 \otimes (f_0 \otimes 1_{D_1})) - 1\,.$$

Here, $\mathrm{cone}(D_0) \subset C([0,1], D_0)$ means the unitization of $C_0((0,1], D_0)$. We denote the natural epimorphism from $\mathrm{cone}(D_0) \otimes^{max} \mathrm{cone}(D_1)$ onto $\mathcal{E}(D_0, D_1)$ by η.

Let $\pi := (\pi_J)_\omega \colon B_\omega \to (B/J)_\omega$ denote the the ultrapower of the epimorphism π_J from B onto B/J. (The kernel of π is J_ω and $\pi(B_\omega) = (B/J)_\omega$.)

Let $A_1 := C^*(A + J)$. Note that $\pi(A_1) = \pi(A) \subset (B/J)_\omega$, $(A_1, B)^c \subset (A,B)^c$, and that $\pi \colon (A_1, B)^c \to (B/J)_\omega$ maps $F(A_1, B) = (A_1, B)^c$ *onto* $(\pi(A), B/J)^c = F(\pi(A), B/J)$ (cf. Remark 1.15). Thus, we can suppose, that $J \subset A \subset B_\omega$.

It suffices to find a unital *-morphism H from $\mathrm{cone}(D_0) \otimes^{max} \mathrm{cone}(D_1)$ into $(A,B)^c = A' \cap B_\omega$ with $H((f_0 \otimes 1) \otimes 1) + H(1 \otimes (f_0 \otimes 1)) = 1$. Below we construct *-homomorphisms $h_1 \colon C_0((0,1], D_1) \to (A,B)^c$ and $h_0 \colon C_0((0,1], D_0) \to (A,B)^c$ with commuting images, such that $h_0(f_0 \otimes 1) + h_1(f_0 \otimes 1) = 1$ and $\pi(h_1(f)) = f(1)$ for all $f \in C_0((0,1], D_1)$. There is a unique unital *-morphism

$$H \colon \ \mathrm{cone}(D_0) \otimes^{max} \mathrm{cone}(D_1) \ \to \ (A,B)^c$$

with $H(g \otimes 1) = h_0(g)$ for all $g \in C_0((0,1], D_0)$ and $H(1 \otimes f) = h_1(f)$ for $f \in C_0((0,1], D_0)$. Then H has the desired property and $\pi(H(1 \otimes f)) = f(1) \in D_1$ for $f \in \mathrm{cone}(D_1)$. The unital *-morphism $h \colon \mathcal{E}(D_0, D_1) \to (A,B)^c = F(A,B)$ with $h \circ \eta = H$ satisfies $\pi(h(f)) = f(1)$ for $f \in \mathrm{cone}(D_1)$.

$J_\omega \cap (A,B)^c$ is a σ-ideal of $(A,B)^c$ (cf. Corollary 1.7) and

$$0 \to A' \cap J_\omega \to (A,B)^c \to (\pi(A), B/J)^c$$

is short-exact and strongly locally liftable (cf. Remark 1.15). By Proposition 1.6, there exists a *-morphism $\varphi \colon C_0((0,1], D_1) \to (A,B)^c$ with $\pi(\varphi(f)) = f(1) \in D_1$ for $f \in C_0((0,1], D_1)$. In particular, $1 - \varphi(f_0 \otimes 1) \in J_\omega$. Let $D_2 := \varphi(C_0((0,1], D_1))$. Then $\varphi(C_0((0,1), D_1)) = J_\omega \cap D_2 \subset J_\omega \cap (A,B)^c$. The unital C^*-subalgebra $G := C^*(A, D_2)$ of B_ω is separable. $J_\omega \cap G$ contains $1 - \varphi(f_0 \otimes 1)$, J, and $\varphi((0,1), D_1) = J_\omega \cap D_2$. Let g_0 a strictly positive element of $J_\omega \cap G$. Since J_ω is a σ-ideal of B_ω (by Corollary 1.7), there is a positive contraction $e \in G' \cap J_\omega$ with $e g_0 = g_0$. Then $eb = be$ for all $b \in G \supset A$ and $ej = j$ for all $j \in J_\omega \cap G \supset J$. In particular, $e \in (A,B)^c$ and $(1-e)(1 - \varphi(f_0 \otimes 1)) = 0$. Since e commutes element-wise with D_2, we can modify φ as follows:

There is a unique *-morphism $h_1 \colon \ C_0((0,1], D_1) \to B_\omega$ with

$$h_1(f_0^n \otimes d) = (1-e)^n \varphi(f_0^n \otimes d)$$

for $d \in D_1$ and $n \in \mathbb{N}$. The *-morphism h_1 maps $C_0((0,1], D_1)$ into $(A,B)^c$ and $\pi(h_1(f)) = f(1) \in D_1$ for $f \in C_0((0,1], D_1)$. Note that $h_1(f_0 \otimes 1) = (1-e)$. Now let $G_1 := C^*(eG, e) \subset J_\omega$. Then e is a strictly positive element of G_1 and is in the center of G_1, because $e \in (G,J)^c$.

By Proposition 1.12 there exists a *-morphism ψ from $C_0((0,1], D_0)$ into $(G_1, J)^c = G_1' \cap J_\omega$ with $\psi(f_0 \otimes 1)b = b$ for all $b \in G_1$, i.e.

$$\theta\colon\ d \in D_0 \mapsto \psi(f_0 \otimes d) + \mathrm{Ann}(G_1, J_\omega) \in F(G_1, J)$$

is a unital *-morphism from D_0 into $F(G_1, J)$. Since $e \in G_1$ commutes with the image of ψ we can modify ψ as follows:
There is a unique *-morphism $h_0\colon\ C_0((0,1], D_0) \to B_\omega$ with

$$h_0(f_0^n \otimes d) = e^n \psi(f_0^n \otimes d) = \rho_{G_1}(\theta(d) \otimes e^n)$$

for $d \in D_0$ and $n \in \mathbb{N}$.

Let $b \in G$ and $d \in D_0$, then

$$b e^n h_2(f_0^n \otimes d) = h_2(f_0^n \otimes d) b e^n = e^n \psi(f_0^n \otimes d) b$$

for all $b \in G$, $n \in \mathbb{N}$. Thus, h_0 maps $C_0((0,1] \otimes D_0)$ into $G' \cap B_\omega$, i.e. the image of h_0 is in $(A,B)^c$ and commutes element-wise with the image of h_1. Furthermore, $h_0(f_0 \otimes 1) = \psi(f_0 \otimes 1)e = e$ because $e \in G_1$. Hence, h_1, h_0 define h (via H) with the stipulated properties. □

C Some Calculations with KTP

For convenience of the reader we add here some calculations that help to verify some of the remarks in Section 4.

Proof (of Remark 4.6). Suppose that $\mathcal{D}$ is self-absorbing. By Proposition 4.4, $\mathcal{D}$ is simple, is nuclear, has a unique tracial state or is purely infinite, and $\mathcal{D} \subset F(\mathcal{D})$.

If $[1] = 0$ in $K_0(\mathcal{D})$ then $\mathcal{D}$ can not have a tracial state. Thus $\mathcal{D}$ is purely infinite and $\mathcal{O}_2$ is unitally contained in $\mathcal{D} \subset F(\mathcal{D})$. Hence, $\mathcal{D} \cong \mathcal{O}_2$ by [23] (or [20, p. 135]).

If $\mathcal{D}$ is tensorially self-absorbing, then $\mathcal{D} \otimes \mathcal{O}_\infty$ is tensorially self-absorbing simple p.i.s.u.n. algebra with $K_*(\mathcal{D} \otimes \mathcal{O}_\infty) = K_*(\mathcal{D})$.

Suppose that $K_0(\mathcal{D}) \neq 0$, that $\mathcal{D}$ is a p.i.s.u.n. algebra and that $\mathcal{D}$ satisfies the KTP, i.e. that with $A = B = \mathcal{D}$ there are (unnaturally) splitting short-exact sequences

$$0 \to \mathrm{Tens}(A, B, \alpha) \to K_\alpha(A \otimes B) \to \mathrm{Tor}(A, B, \alpha) \to 0$$

for $A = B = \mathcal{D}$ and $\alpha \in \{0, 1\}$. Here

$$\mathrm{Tens}(A, B, \alpha) := (K_\alpha(A) \otimes K_0(B)) \oplus (K_{1-\alpha}(A) \otimes K_1(B))$$

and,

$$\mathrm{Tor}(A, B, \alpha) := \mathrm{Tor}(K_0(A), K_{1-\alpha}(B)) \oplus \mathrm{Tor}(K_1(A), K_\alpha(B)).$$

The monomorphism $K_\alpha(\mathcal{D})\otimes K_0(\mathcal{D}) \to K_\alpha(\mathcal{D}\otimes\mathcal{D})$ is induced by $[x]_\alpha\otimes[p]_0 \mapsto [x\otimes p]_\alpha$ for projections $p\in\mathcal{D}$ and projections or unitaries in $\mathcal{D}$.

The isomorphisms $K_\alpha(\mathcal{D})\otimes[1_\mathcal{D}]_{K_0} \cong K_\alpha(\mathcal{D}\otimes\mathcal{D})$ imply that $K_1(\mathcal{D})\otimes K_1(\mathcal{D}) = 0$, and that $\mathrm{Tor}(K_\alpha(\mathcal{D}), K_\alpha(\mathcal{D})) = 0$ for $\alpha = 0,1$. Thus $K_0(\mathcal{D})$ and $K_1(\mathcal{D})$ are torsion-free (because all Abelian groups with a non-zero torsion element have some $\mathbb{Z}_p$ or some p-Prüfer-group $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ as direct summand, cf. [16, cor. 27.3]). Therefore, $K_1(\mathcal{D})\otimes K_1(\mathcal{D}) = 0$ implies $K_1(\mathcal{D}) = 0$. The flip on $\mathcal{D}\otimes\mathcal{D}$ induces the flip on $K_0(\mathcal{D})\otimes K_0(\mathcal{D}) \cong K_0(\mathcal{D}\otimes\mathcal{D})$, i.e. $[1]\otimes_\mathbb{Z} x = x\otimes_\mathbb{Z}[1]$ in $K_0(\mathcal{D})\otimes K_0(\mathcal{D})$ for $x\in K_0(\mathcal{D})$. This means that there are non-zero $m,n\in\mathbb{Z}$ with $m[1] = nx$. Thus $K_0(\mathcal{D})$ is a unital subring of the rational numbers $\mathbb{Q}$ (if $K_0(\mathcal{D})\neq 0$).
If we now suppose in addition that $\mathcal{D}$ satisfies the UCT, then the classification of simple p.i.s.u.n. algebras yields $\mathcal{D} = \mathcal{O}_\infty$ if $K_0(\mathcal{D})\cong\mathbb{Z}$, and

$$\mathcal{D} = \mathcal{O}_\infty \otimes (\bigotimes_{p\in X} M_{p^\infty})\,,$$

where X is the set of prime numbers with $1/p\in K_0(\mathcal{D})\subset\mathbb{Q}$ if $\mathcal{D}\ncong\mathcal{O}_2,\mathcal{O}_\infty$.

If $\mathcal{D}$ is not purely infinite, then $\mathcal{D}$ has a unique tracial state τ, and τ defines an order preserving isomorphism from $K_0(\mathcal{D})$ onto the subring $\tau(K_0(\mathcal{D}))$ of the rational numbers. It is an order isomorphism if and only if $(K_0(\mathcal{D}), K_0(\mathcal{D})_+)$ is weakly unperforated.

Thus, the given list of algebras exhausts all possible Elliott invariants that could appear for the algebras $\mathcal{D}\otimes\mathcal{Z}$ in the UCT-class. □

Proof (of Remarks 4.3). (1): The Cuntz algebra $\mathcal{O}_2$ is isomorphic to $\mathcal{D} := D\otimes D\otimes\ldots$, because $\mathcal{O}_2$ is unitally contained in $F(\mathcal{D})$ (cf. proof of Remark 2.17). Since $\mathcal{O}_2\cong\mathcal{O}_2\otimes D$, η_1 and η_2 map D into (different) unital copies of of $\mathcal{O}_2$ in $D\otimes D$. Thus $[\eta_1] = 0 = [\eta_2]$ in $KK(D, D\otimes D)$. It implies that η_1 and η_2 are approximately unitarily equivalent (they even are unitarily homotopic by a basic result of classification).

(2): a) $\mathcal{P}_\infty\otimes\mathcal{P}_\infty$ is stably isomorphic to to $\mathcal{O}_\infty$ (by the classification theorem for simple p.i.s.u.n. algebras in the UCT-class and by the KTP).

b) The unit of $\mathcal{O}_\infty$ is Murray–von-Neumann equivalent to the the Bott projection $p(U\otimes 1, 1\otimes U)\in M_2(\mathcal{P}_\infty\otimes\mathcal{P}_\infty)$ (defined below) from a unitary $U\in\mathcal{P}_\infty$ such that $[U] = 1$ in $\mathbb{Z}\cong K_1(\mathcal{P}_\infty)$. This follows from the KTP and the definition of the isomorphism $K_1(\mathcal{P}_\infty)\otimes K_1(\mathcal{P}_\infty)\cong K_0(\mathcal{P}_\infty\otimes\mathcal{P}_\infty)$ in the KTP.

c) The K_0-class of a Bott projection $p(V,W)$ for commuting unitaries V,W reverses its sign if V and W will be interchanged:
Let V,W commuting unitaries in a unital algebra B, and let $h_{V,W}$ denote the *-morphism from $C(S^1)\otimes C(S^1)$ into B with $h_{V,W}(u_0\otimes 1) = V$ and $h_{V,W}(1\otimes u_0) = W$. The Bott projection $p(V,W)\in M_2(B)$ is the image $h_{V,W}\otimes\mathrm{id}_2(p_{\mathrm{Bott}})\in M_2(C^*(V,W))\subset M_2(B)$ of the canonical Bott projection $p_{\mathrm{Bott}}\in M_2(C(S^1)\otimes C(S^1))$.

p_{Bott} is contained in the unital subalgebra $(C_0(\mathbb{R})\otimes C_0(\mathbb{R}))+\mathbb{C}\cdot 1\cong C(S^2)$ of $(C_0(\mathbb{R})+\mathbb{C}1)\otimes(C_0(\mathbb{R})+\mathbb{C}1)\cong C(S^1)\otimes C(S^1)$ and $[p_{\mathrm{Bott}}]-[1\otimes e_{1,1}]$

generates $K_0(C_0(\mathbb{R}^2)) \cong \mathbb{Z}$. Let $D := \{z \in \mathbb{C}\,;\ |z| \leq 1\}$ the closed unit disk in $\mathbb{C}$, $S^1 = \partial D$ its boundary and $\psi\colon\ z = x + iy \in \mathbb{C} \cong \mathbb{R}^2 \mapsto (1 + |z|^2)^{-1/2} z \in D$ the natural homeomorphism from $\mathbb{R}^2$ onto $D \setminus S^1$. The 6-term exact K_*-sequence of the corresponding exact sequence

$$0 \to C_0(\mathbb{R}) \otimes C_0(\mathbb{R}) \to C(D) \to C(S^1) \to 0$$

defines a boundary isomorphism ∂ from $K_1(C(S^1))$ onto $K_0(C_0(\mathbb{R}) \otimes C_0(\mathbb{R}))$. This isomorphism is functorial with respect to *-morphisms $\widehat{\chi}$ of $C(S^1)$ respectively of $C_0(\mathbb{R}) \otimes C_0(\mathbb{R})$ that are induced by continuous maps χ from (D, S^1) into (D, S^1).

The flip $(x, y) \in \mathbb{R}^2 \mapsto (y, x) \in \mathbb{R}^2$ is induced by $\psi^{-1}(\chi(\psi(x + iy)))$, where χ is the homeomorphism of D given by $\chi(w) := i\overline{w}$ for $w \in D$. The homeomorphism $\chi|S^1$ reverses the orientation of S^1, hence

$$K_1(\widehat{\chi|S^1})\colon\ K_1(C(S_1)) \to K_1(C(S_1))$$

is the isomorphism $n \mapsto -n$ of $K_1(C(S_1)) \cong \mathbb{Z}$. Therefore, the flip automorphism of $C_0(\mathbb{R}) \otimes C_0(\mathbb{R})$ defines the automorphism of $K_0(C_0(\mathbb{R}^2)) \cong \mathbb{Z}$ that changes signs. The restriction of $h_{V,W}$ to $C_0(\mathbb{R}) \otimes C_0(\mathbb{R})$ defines a group morphism $\mu_{V,W}$ from $\mathbb{Z} \cong K_0(C_0(R^2))$ to $K_0(C^*(V, W))$ (and then to $K_0(B)$ for commuting unitaries $V, W \in B$) with $\mu_{V,W}(1) = [p_{V,W}] - [1 \otimes e_{1,1}]$.

d) By a) and c), the flip map on $\mathcal{P}_\infty \otimes \mathcal{P}_\infty \cong (\mathcal{O}_\infty)^{st}$ defines an automorphism of $\mathcal{O}_\infty \otimes \mathcal{K}$ of order 2 that reverses the sign of elements $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$.

In particular, the flip of $\mathcal{P}_\infty \otimes \mathcal{P}_\infty$ is *not* approximately inner.

(3): The examples of Rørdam are not stably finite. □

Proof (of Remark 4.2). Let $\delta_n(d) := \sum_{1 \leq j \leq n} s_j d s_j^*$ for $d \in \mathcal{O}_n$ and the canonical generators $s_1, \ldots, s_n$ of $\mathcal{O}_n$. Since $\delta_n\colon \mathcal{O}_n \to \mathcal{O}_n$, is unital and is homotopic to id, δ_n is approximately unitarily equivalent to id (by classification theory). Thus, $\mathcal{O}_n$ is unitally contained in $F(\mathcal{O}_n)$. By Corollary 1.13 this implies that $\mathcal{D} := \mathcal{O}_n \otimes \mathcal{O}_n \otimes \cdots$ is unitally contained in $F(\mathcal{O}_n)$. Since $\mathcal{O}_n \not\cong \mathcal{O}_2$ we get that $\mathcal{O}_2$ is not unitally contained in $\mathcal{D}$, i.e. $0 \neq [1] \in K_0(\mathcal{D})$ (cf. proof of 2.17). Moreover, $\mathcal{D}$ is a p.i.s.u.n. algebra in the UCT-class and $(n-1)K_*(\mathcal{D}) = \{0\}$, because $\mathcal{O}_n$ is a p.i.s.u.n. algebra in the UCT class and $\mathcal{D} \cong \mathcal{O}_n \otimes \mathcal{D}$.

Suppose that $\eta_{1,\infty}$ and $\eta_{2,\infty}$ are approximately unitarily equivalent in $\mathcal{D}$. Then $\mathcal{D}$ is self-absorbing by Corollary 4.12. Since $(n-1)K_*(\mathcal{D}) = \{0\}$, Remark 4.6 implies that $K_*(\mathcal{D}) \cong 0$, which contradicts $0 \neq [1] \in K_0(\mathcal{D})$. □

References

1. C.A. Akemann, G.K. Pedersen, *Central sequences and inner derivations of separable C*-algebras*, Amer. J. Math. **101** [1979], 1047–1061.
2. W. Arveson, *Notes on extensions of C*-algebras*, Duke Math. J. **44** [1977], 329–355.

3. B. Blackadar, *K-theory for Operator Algebras*, second edition, MSRI Publications **5**, Cambridge Univ. Press [1998].
4. B. Blackadar, E. Kirchberg, *Inner quasidiagonality and strong NF algebras*, Pacific J. Math. **198** [2001], 307–329.
5. E. Blanchard, E. Kirchberg, *Global Glimm halving for C^*-bundles*, J. Operator Theory **52** [2004], 385–420.
6. E. Blanchard, E. Kirchberg, *Non-simple purely infinite C^*-algebras: the Hausdorff case*, J. Funct. Analysis **207** [2004], 461–513.
7. L.G. Brown, *Stable isomorphism of hereditary subalgebras of C^*-algebras*, Pacific J. Math. **71** [1977], 335–384.
8. A. Connes, *Classification of injective factors*, Ann. of Math. **104** . [1976], 73–115.
9. J. Cuntz, *Dimension functions on simple C^*-algebras*, Math. Ann. **233** [1978], 145–153.
10. J. Cuntz, *K-theory for certain C^*-algebras*, Ann. of Math. **113** [1981], 181–197.
11. J. Cuntz, *The internal structure of simple C^*-algebras*, Proc. Symp. Pure Math. (R.V. Kadison, ed.) **38**, Part 1, AMS, Providence, R.I. [1982], 85–115.
12. J. Cuntz, *K-theory for certain C^*-algebras II*, J. Operator Theory **5** [1981], 101–108.
13. W.W. Comfort, Negrepontis, S. [1974], *The theory of ultrafilters*, Grundlehren der math. Wiss. **211**, Springer.
14. K.J. Dykema, M. Rørdam, *Purely infinite simple C^*-algebras arising from free product constructions*, Canad. J. Math. **50** [1998], 323-341.
15. E. Effros, J. Rosenberg, *C^*-algebras with approximately inner flip*, Pacific J. Math. **77** [1978], 417–443.
16. L. Fuchs, *Infinite Abelian groups*, Volume I, Academic Press, New York and London [1970].
17. K. Grove, G.K. Pedersen, *Sub-Stonean spaces and corona sets* J. Funct. Analysis **56** [1984], 124–143.
18. X. Jian, H. Su, *On a simple projectionless C^*-algebra*, Amer. J. Math. **121** [1999], 359–413.
19. E. Kirchberg, *On restricted perturbations in inverse images and a description of normalizer algebras in C^*-algebras*, J. Funct. Analysis **129** [1995], 1–34.
20. E. Kirchberg, *Das nicht-kommutative Michael-Auswahlprinzip und die Klassifikation nicht-einfacher Algebren*, in *C^*-Algebras*: Proceedings of the SFB-Workshop on C^*-algebras, Münster, Germany, March 8-12, 1999/ J. Cuntz, S. Echterhoff (ed.), Berlin etc., Springer [2000], 272 pp.
21. E. Kirchberg, *The classification of purely infinite C^*-algebras using Kasparov theory*, notes (expanded english version of [20] and of the 3rd draft of the preprint with the same title from Dec 1994).
22. E. Kirchberg, *Permanence properties of strongly purely infinite C^*-algebras*, SFB478-preprint, Univ. Münster.
23. E. Kirchberg, C.E. Phillips, *Embedding of exact C^*-algebras in the Cuntz algebra $\mathcal{O}_2$*, J. reine angew. Math. **525** [2000], 17–53.
24. E. Kirchberg, M. Rørdam, *Non-simple purely infinite C^*-algebras*, Amer. J. Math. **122** [2000], 637–666.
25. E. Kirchberg, M. Rørdam, *Infinite non-simple C^*-algebras: absorbing the Cuntz algebra $\mathcal{O}_\infty$*, Advances in Math. 167 [2002], 195–264.
26. E. Kirchberg, W. Winter, *Covering dimension and quasi-diagonality*, International J. of Math. 14 [2003], 1–23.

27. T. Loring, *Lifting Solutions to Perturbing Problems in C^*-algebras*, Field Institute monographs, vol.**8**, Amer. Math. Soc., Providence, Rhode Island, [1997].
28. T. Loring, G.K. Pedersen *Projectivity, transitivity and AF-telescopes* Trans. Amer. Math. Soc. **350** [1998], 4313-4339.
29. G.K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London [1979].
30. M. Rørdam, *A simple C^*-algebra with a finite and an infinite projection*, Acta Mathematica **191** [2003], 109–142.
31. M. Rørdam, *The stable rank and the real rank of $\mathcal{Z}$-absorbing C^*-algebras*, arXiv-preprint [Aug 2004].
32. A. Toms, and W. Winter, *Strongly self-absorbing C^*-algebras*, Uni. Münster, SFB478-preprint, Heft 366, Febr. 2005.
33. S. Zhang, *A property of purely infinite simple C^*-algebras*, Proc. Amer. Math. Soc. **109** [1990], 717–720.

Lifting of an Asymptotically Inner Flow for a Separable C^*-Algebra

Akitaka Kishimoto

Department of Mathematics, Hokkaido University, Sapporo, Japan
kishi@math.sci.hokudai.ac.jp

1 Introduction

Before this symposium I had been pondering over approximately inner flows to see whether I could crack something with a new tool or two I might find in my toolbox. My approach was rather classical as usual; so the problem was to explore such flows in relation with invariant hereditary C*-subalgebras, extensions, tensor products, etc. To my disappointment I hardly got anything. Assuming nobody else has tried recently, the present knowledge on these flows does not seem to exceed much what is already presented in Bratteli-Robinson's book [2, 3] and Sakai's book [15]; notable results there are concerned with KMS states and representations in addition to a broad theory of unbounded derivations and generators and a theory in AF algebras.

My favorite (and only pertinent) result I had at that time was an *existence* result of approximately inner flows [8], which was obtained at the same time as the existence result of single automorphisms was in [12]. After the symposium I got a *lifting* theorem, which partly generalizes results by Pedersen, Olesen, and Elliott for universally weakly inner flows, referred to by Olesen at the conference (see [13, 4]). But to prove this lifting theorem, I have to introduce a class of asymptotically inner flows in parallel with the case of single automorphisms; the result would say such a flow can be lifted from a quotient of a separable C*-algebra.

Without giving the definition precisely, I would say that all the known examples of approximately inner flows are actually asymptotically inner. In the next section I will give a few comments on this new notion and report on the existence result with some details. In section 3 I will then discuss the lifting theorem for flows. I will also add a similar result for automorphisms since I believe this has not been presented yet.

There was another type of flows I reported on in my talk, namely, the Rohlin flows, which are far from the asymptotically inner flows but could be more manageable by its strong property of cocycle vanishing (at least when the C*-algebra is a Kirchberg algebra [5, 6]). I will not discuss it here (for interested readers see [9, 10, 11]).

I conclude this introduction by giving some basics on flows [2, 15] and the definition of asymptotically inner flows.

By a flow α on a C*-algebra A we mean a homomorphism $\alpha : \mathbf{R} \to \mathrm{Aut}(A)$ such that $t \mapsto \alpha_t(x)$ is continuous for each $x \in A$, where $\mathrm{Aut}(A)$ is the automorphism group of A. When α is a flow, we denote by δ_α the generator of α, which is a closed derivation in A, i.e., δ_α is a closed linear map defined on a dense $*$-subalgebra $D(\delta_\alpha)$ of A into A such that $\delta_\alpha(x)^* = \delta_\alpha(x^*)$ and

$$\delta_\alpha(xy) = \delta_\alpha(x)y + x\delta_\alpha(y), \quad x, y \in D(\delta_\alpha).$$

Moreover δ_α is well-behaved, or $\pm\delta_\alpha$ is dissipative. (But a well-behaved closed derivation need not be a generator.) Note that $D(\delta_\alpha)$ is a Banach $*$-algebra with the norm defined by embedding $D(\delta_\alpha)$ into $M_2(A)$ by

$$x \mapsto \begin{pmatrix} x & \delta_\alpha(x) \\ 0 & x \end{pmatrix}.$$

Given $h \in A_{sa}$, $\delta_\alpha + \mathrm{ad}\, ih$ is again a generator, where $\mathrm{ad}\, ih(x) = i(hx - xh)$, $x \in A$. We denote by $\alpha^{(h)}$ the flow generated by $\delta_\alpha + \mathrm{ad}\, ih$. We call $\alpha^{(h)}$ an inner perturbation of α. More generally, if u is an α-cocycle, i.e., $u : \mathbf{R} \to \mathcal{U}(A)$ is continuous such that $u_s\alpha_s(u_t) = u_{s+t}$, $s, t \in \mathbf{R}$, then $t \mapsto \mathrm{Ad}\, u_t\alpha_t$ is a flow, called a cocycle perturbation of α. Note that an inner perturbation is a cocycle perturbation; $\alpha^{(h)}$ is obtained as $\mathrm{Ad}\, u^{(h)}\alpha$, where $u = u^{(h)}$ is the (differentiable) α-cocycle defined by $du_t/dt = u_t\alpha_t(ih)$ and $u_0 = 1$. In general a cocycle perturbation of α is given as $t \mapsto \mathrm{Ad}\, v\alpha_t^{(h)}\mathrm{Ad}\, v^* = \mathrm{Ad}(vu_t^{(h)}\alpha_t(v^*))\alpha_t$ for some $v \in \mathcal{U}(A)$ and $h \in A_{sa}$.

We will use the following result below (see [2, 15]).

Proposition 1. *Let α be a flow on a C*-algebra A and let (h_n) be a sequence in A_{sa}. Then the following conditions are equivalent.*

1. $\lim_{n\to\infty} \max_{|t|\le 1} \|\alpha_t(x) - \mathrm{Ad}\, e^{ith_n}(x)\| = 0,\ x \in A.$
2. $\delta_\alpha = \lim_{n\to\infty} \mathrm{ad}\, ih_n$ *in the graph sense, i.e., (in this case) for any* $x \in D(\delta_\alpha)$ *there is a sequence* (x_n) *in* A *such that* $\lim_{n\to\infty} \|x - x_n\| = 0$ *and* $\lim_{n\to\infty} \|\delta_\alpha(x) - \mathrm{ad}\, ih_n(x_n)\| = 0$.

The flow α as in the above proposition is called an *approximately inner* flow. Let us define *asymptotically inner* flows in the same way as we do asymptotically inner automorphisms for approximately inner automorphisms.

Definition 2. *A flow α on a C*-algebra A is said to be* asymptotically inner *if there is a continuous function h of $\mathbf{R}_+$ into A_{sa} such that*

$$\lim_{s\to\infty} \max_{|t|\le 1} \|\alpha_t(x) - \mathrm{Ad}\, e^{ith(s)}(x)\| = 0$$

for any $x \in A$. In this case we say that $\mathrm{Ad}\, e^{ith(s)}$ *converges to* α_t *as* $s \to \infty$ *or* α_t *is the limit of* $\mathrm{Ad}\, e^{ith(s)}$ *as* $s \to \infty$.

It is obvious that asymptotical innerness implies approximate innerness. We know, for single automorphisms, that the class of asymptotically inner automorphisms is smaller than the class of approximately inner automorphisms in general (see [5, 6]).

The following is an easy corollary of what is shown in [4, 13] and gives a class of asymptotically inner flows.

Proposition 3. *Let A be a separable C*-algebra and let α be a flow on A. If α is universally weakly inner (i.e., there is a weak*-continuous unitary flow u in A^{**} such that $\alpha_t(x) = \mathrm{Ad}\, u_t(x),\ x \in A$), then it is asymptotically inner.*

If A is unital and simple, then a universally weakly inner flow is just uniformly continuous and hence is inner.

2 Asymptotically Inner Flows

With the two similar properties for flows at hand, I suppose we must say something about them. What I have to confess is that I do not know whether or not the notion of asymptotical innerness is strictly stronger than the one of approximate innerness. But a naive expectation would fail. For example, if α is a flow on A and (h_n) is a sequence in A_{sa} such that α_t is the limit of $\mathrm{Ad}\, e^{ith_n}$, we may expect, defining $h : \mathbf{R}_+ \to A_{sa}$ by $h(s) = (n-s)h_{n-1} + (s-n+1)h_n,\ s \in [n-1, n]$, that α_t is the limit of $\mathrm{Ad}\, e^{ith(s)}$ as $s \to \infty$, which is not the case in general as shown by the following simple example.

Example 4. Let A be a unital simple AF algebra and let (A_n) be an increasing sequence of finite-dimensional C*-subalgebras of A such that $\bigcup_n A_n$ is dense in A. Let $h, a \in (A_1)_{sa}$ be such that $[h, a] \neq 0$. Let $x_n, y_n \in (A \cap A_n')_{sa}$ be such that $\|y_n\| = 1$ and $\|[x_n, y_n]\| \to \infty$ (and so $\|x_n\| \to \infty$). Let $\epsilon_n = \|[x_n, y_n]\|^{-1}$ and define

$$h_{2n-1} = e^{i\epsilon_n h y_n} x_n e^{-i\epsilon_n h y_n} \approx x_n + i\epsilon_n h[y_n, x_n]$$

(with an error of order ϵ_n) and $h_{2n} = -x_n$.

Then $\mathrm{Ad}\, e^{ith_n}$ converges to the trivial flow id since $e^{i\epsilon_n h y_n}$ converges to 1 and $\mathrm{Ad}\, e^{itx_n}$ converges to id. Let

$$k_n = \frac{1}{2}(h_{2n-1} + h_{2n}) \approx \frac{1}{2} i\epsilon_n h[y_n, x_n].$$

It then follows that (k_n) is a bounded sequence and that $(\mathrm{ad}\, ik_n)$ does not converge since $\mathrm{ad}\, ik_n(a) \approx i[h, a]\epsilon_n[y_n, x_n]$. This implies that $\mathrm{Ad}\, e^{itk_n}$ does not converge.

We often encounter the situation given in the following proposition:

Proposition 5. *Let α be a flow on a C^*-algebra A. Suppose that there is a sequence (h_n) in A_{sa} and a $*$-subalgebra D of A such that $D \subset D(\delta_\alpha)$, D is a core for δ_α, and*

$$\delta_\alpha(x) = \lim_{n \to \infty} \mathrm{ad}\, ih_n(x), \quad x \in D.$$

Then α is asymptotically inner.

Proof. In this situation we define a continuous function $h : \mathbf{R}_+ \to A_{sa}$ by

$$h(s) = (n-s)h_{n-1} + (s-n+1)h_n, \quad s \in [n-1, n]$$

with $h_0 = 0$. Then it follows that $\delta_\alpha(x) = \lim_{s \to \infty} \mathrm{ad}\, ih(s)(x),\ x \in D$ and that δ_α is the graph limit of $\mathrm{ad}\, ih(s)$ as $s \to \infty$. From this the conclusion follows.

The core condition for D above may not be possible to prove. Another situation we may encounter is as follow:

Proposition 6. *Let α be a flow on a C^*-algebra A. Suppose that there is a sequence (h_n) in A_{sa} and a dense $*$-subalgebra D of A such that δ_α is the graph limit of $(\mathrm{ad}\, ih_n)$, $h_n \in D \subset D(\delta_\alpha)$,*

$$\delta_\alpha(x) = \lim_{n \to \infty} \mathrm{ad}\, ih_n(x), \quad x \in D,$$

and $(\|\delta_\alpha(h_n)\|)$ is bounded. Then α is asymptotically inner.

Proof. We define a continuous function $h : \mathbf{R}_+ \to A_{sa}$ by linearly interpolating $n \mapsto h_n$ as in the previous proof. Then it follows that for any increasing sequence (s_n) in $\mathbf{R}_+$ such that $s_n \to \infty$, the sequence $(h(s_n))$ satisfies the same conditions as (h_n) does, except that δ_α is the graph limit of $(\mathrm{ad}\, ih(s_n))$.

Suppose that (s_n) includes $\mathbf{N}$ as a subsequence. Let δ be the graph limit of $(\mathrm{ad}\, ih(s_n))$. Then δ is a restriction of δ_α such that $D(\delta) \supset D$. It follows from 3.1 of [1] that δ is a generator, i.e., $\delta = \delta_\alpha$. Thus we can conclude that α is asymptotically inner.

A flow α is an *asymptotically inner perturbation* of a flow β if there is a continuous function $h : \mathbf{R}_+ \to A_{sa}$ such that $\mathrm{Ad}\, \beta_t^{(h(s))}$ converges to α_t, i.e.,

$$\lim_{s \to \infty} \max_{|t| \le 1} \|\alpha_t(x) - \beta_t^{(h(s))}(x)\| = 0$$

for any $x \in A$.

With this definition, an asymptotically inner flow is an asymptotically inner perturbation of the trivial flow id. Then there arises a natural problem: If α is an asymptotically inner flow, is the trivial flow id an asymptotically inner perturbation of α? Although this looks quite plausible, I am embarrassed to say that I do not know the answer. But again a naive expectation would fail: If α_t is the limit of $\mathrm{Ad}\, e^{ith(s)}$ as $s \to \infty$, $\alpha^{(-h(s))}$ need not converge to id as $s \to \infty$ as shown as follows:

Example 7. Let A be a unital simple AF algebra and let α be a flow on A such that there is an increasing sequence (A_n) of unital finite-dimensional C*-subalgebras of A such that $A = \overline{\bigcup_n A_n}$ and $\bigcup_n A_n \subset D(\delta_\alpha)$. Suppose that $\bigcup_n A_n$ is a core for δ_α and δ_α is unbounded. Then there is a continuous function $h : \mathbf{R}_+ \rightarrow A_{sa}$ such that δ_α is the graph limit of $\mathrm{ad}\, ih(s)$ as $s \rightarrow \infty$ but $\delta_\alpha - \mathrm{ad}\, ih(s)$ does not converge to zero in the graph sense as $s \rightarrow \infty$.

In the above situation there is a $k_n \in A_{sa}$ such that $\delta_\alpha | A_n = \mathrm{ad}\, ik_n | A_n$. Then since $\bigcup_n A_n$ is a core for δ_α, we get that $\mathrm{ad}\, ik_n$ converges to δ_α as $n \rightarrow \infty$ in the graph sense. Moreover, since $\delta_\alpha - \mathrm{ad}\, ik_n$ converges to zero on each element of $\bigcup_n A_n$, we get that $\delta_\alpha - \mathrm{ad}\, ik_n$ converges to zero in the graph sense. By passing to a subsequence of (A_n) and by giving a small inner perturbation to δ_α, we may suppose that $\delta_\alpha(A_n) \subset A_{n+1}$.

Assume that A_1 is not commutative and fix $a, x \in (A_1)_{sa}$ such that $\|a\| = 1$ and $[a, x] \neq 0$. We find a sequence (b_n) in A_{sa} and a sequence (ℓ_n) in $\mathbf{N}$ such that $b_n \in A_n \cap A'_{\ell_n}$, $\|b_n\| = 1$, $\|\delta_\alpha(b_n)\| \rightarrow \infty$, $n \geq \ell_n$, and $\ell_n \rightarrow \infty$. We set $u_n = e^{i\epsilon_n a b_n}$, where $\epsilon_n = \max\{1, \|\delta_\alpha(b_n)\|\}^{-1}$. We define $h_n = u_n k_{n+1} u_n^*$. Since $u_n \rightarrow 1$ and

$$\mathrm{Ad}\, e^{ith_n}(y) = u_n \mathrm{Ad}\, e^{itk_{n+1}}(u_n^* y u_n) u_n^*,$$

$\mathrm{Ad}\, e^{ith_n}$ converges to α_t as $n \rightarrow \infty$ or $\mathrm{ad}\, ih_n$ converges to δ_α in the graph sense.

We can interpolate (h_n), i.e., we have a continuous function $h : \mathbf{R}_+ \rightarrow A_{sa}$ such that $h(n) = h_n$ (and perhaps taking on k_{n+1} after and before n) and $\mathrm{ad}\, ih(s)$ converges to δ_α in the graph sense.

We assert that $\delta_\alpha - \mathrm{ad}\, ih(s)$ does not converge to zero (in the graph sense). For this purpose it suffices to show that $\delta_\alpha - \mathrm{ad}\, ih_n$ does not converge to zero.

Suppose that $\delta_\alpha - \mathrm{ad}\, ih_n$ converge to zero, which implies that $\mathrm{Ad}\, u_n^*(\delta_\alpha - \mathrm{ad}\, ih_n)\mathrm{Ad}\, u_n = \delta_\alpha + \mathrm{ad}\, u_n^* \delta_\alpha(u_n) - \mathrm{ad}\, ik_{n+1}$ also converge to zero.

Since $\|\delta_\alpha(u_n) - i\epsilon_n a \delta_\alpha(b_n)\| \rightarrow 0$, we get that

$$\delta_\alpha - \mathrm{ad}\, ik_{n+1} + \mathrm{ad}\, i\epsilon_n a \delta_\alpha(b_n) \rightarrow 0.$$

For the $x \in A_1$ we have chosen before, we get a sequence (x_n) in $D(\delta_\alpha)$ such that $\|x - x_n\| \rightarrow 0$ and

$$\delta_\alpha(x_n) - \mathrm{ad}\, ik_{n+1}(x_n) + i\epsilon_n [a, x_n] \delta_\alpha(b_n) \rightarrow 0,$$

where we have used that $\delta_\alpha(b_n) \in A_{n+1} \cap A'_{\ell_n - 1}$. We will show that this is absurd.

Let β denote the flow generated by $\delta_\alpha - \mathrm{ad}\, ik_{n+1}$. Then $\beta_t | A_{n+1} = \mathrm{id}$. Note that $[a, x] \in A_1$ and $\delta_\alpha(b_n) \in A_{n+1} \cap A'_{\ell_n - 1}$ as asserted above. Since A is simple, we get that $\|[a, x]\delta_\alpha(b_n)\| = \|[a, x]\| \cdot \|\delta_\alpha(b_n)\| \neq 0$ for all large n. Let ϕ_n be a state of A_{n+1} such that $|\phi_n(\epsilon_n [a, x] \delta_\alpha(b_n))| = \|[a, x]\|$ for such n. Let $\overline{\phi}_n$ be an extension of ϕ_n to a state of A and let ψ_n be an average of $\overline{\phi}_n \beta_t$

over $t \in \mathbf{R}$. Then, since $\psi_n \circ (\delta_\alpha - \mathrm{ad}\, ik_{n+1}) = 0$ and $\psi_n | A_{n+1} = \phi_n$, we get that

$$\begin{aligned} &|\psi_n(\delta_\alpha(x_n) - \mathrm{ad}\, ik_{n+1}(x_n) + \mathrm{ad}\, i\epsilon_n[a, x_n]\delta_\alpha(b_n))| \\ \to &\lim_{n\to\infty} |\psi_n(i\epsilon_n[a, x]\delta_\alpha(b_n))| = \|[a, x]\|, \end{aligned}$$

which is a contradiction.

We note the following result.

Proposition 8. *Suppose that the C^*-algebra is unital. The following statements hold.*

1. *A cocycle perturbation of an asymptotically inner flow is asymptotically inner.*
2. *An asymptotically inner perturbation of an asymptotically inner flow is asymptotically inner.*

Proof. Let α be an asymptotically inner flow on A and let $h : \mathbf{R}_+ \to A_{sa}$ be such that α_t is the limit of $\mathrm{Ad}\, e^{ith(s)}$ as $s \to \infty$.

If $b \in A_{sa}$, let u_t denote the α-cocyle such that $du_t/dt|_{t=0} = ib$ and let $u_t^{(s)}$ denote the $\mathrm{Ad}\, e^{ith(s)}$-cocycle such that $du_t^{(s)}/dt|_{t=0} = ib$. Then, by the explicit series expansions of u_t and $u_t^{(s)}$, we have that

$$\lim_{s\to\infty} \max_{|t|\le 1} \|u_t - u_t^{(s)}\| = 0.$$

Thus it follows that $\alpha_t^{(b)} = \mathrm{Ad}\, u_t \alpha_t$ is obtained as the limit of $\mathrm{Ad}\, e^{it(h(s)+b)} = \mathrm{Ad}\, u_t^{(s)} \mathrm{Ad}\, e^{ith(s)}$.

If $z \in A$ is a unitary, there is a unitary $w \in D(\delta_\alpha)$ such that $\|z - w\| < 2$. We express $zw^* = e^{ih}$ for some $h \in A_{sa}$ and find a continuous function $k : \mathbf{R}_+ \to D(\delta_\alpha) \cap A_{sa}$ such that $k(0) = 0$ and $\lim_{s\to\infty} \|k(s) - h\| = 0$ (where we assume that $s \mapsto \delta_\alpha(k(s))$ is continuous as well as $s \mapsto k(s)$). Namely, by taking $s \mapsto e^{ik(s)}w$, we find a continuous function $v : \mathbf{R}_+ \to D(\delta_\alpha) \cap \mathcal{U}(A)$ such that $\lim_{s\to\infty} \|z - v(s)\| = 0$. Define a continuous function $h \to A_{sa}$ by $h(s) = -iv(s)\delta_\alpha(v(s)^*)$. Then $\alpha_t^{(h(s))}$ converges to $\mathrm{Ad}\, z\alpha_t \mathrm{Ad}\, z^*$ (although $\|h(s)\| \to \infty$ if $z \notin D(\delta_\alpha)$). This completes the proof of (1) (by using (2) below) since any α-cocycle is given as $zu_t\alpha_t(z^*)$ with u_t differentiable.

Furthermore if $k : \mathbf{R}_+ \to A_{sa}$ is continuous and a flow β is obtained as the limit of $\alpha^{(k(s))}$ with α as above, one can easily see that for any $x \in A$, the continuous function $[-1, 1] \ni t \mapsto \beta_t(x)$ can be approximated by $[-1, 1] \ni t \mapsto \alpha_t^{(k(s))}(x)$ for large $s \in \mathbf{R}_+$ and then approximated by $[-1, 1] \ni t \mapsto \mathrm{Ad}\, e^{it(h(\sigma)+k(s))}(x)$ (from the first part of the proof of (1)), where σ should be large depending on s. In this way we find a continuous function $\sigma : s \to \mathbf{R}_+$ such that $e^{it(k(\sigma(s))+h(s))}$ converges to β_t. This completes the proof of (2).

We note the following easy implication; we could not prove the converse.

Proposition 9. *Let A be a separable C^*-algebra and I an ideal of A. Let α be an asymptotically inner flow on A. Then it follows that α leaves I invariant, the restriction $\alpha|I$ is asymptotically inner, and so is the quotient $\dot{\alpha}|A/I$.*

Proof. Let $h : \mathbf{R}_+ \to A_{sa}$ be a continuous function such that α_t is the limit of $\mathrm{Ad}\, e^{ith(s)}$ as $s \to \infty$. If Q denotes the quotient map from A onto A/I, then $\mathrm{Ad}\, e^{itQ(h(s))}$ converges to $\dot{\alpha}_t$ on A/I. Hence $\dot{\alpha}$ is asymptotically inner.

For $x \in D(\delta_\alpha) \cap I = D(\delta_{\alpha|I})$, let $x(s) = (1 + \mathrm{ad}\, ih(s))^{-1}(1 + \delta_\alpha)(x)$. It then follows that $x(s) \in I$, $\|x(s) - x\| \to 0$, and $\|\mathrm{ad}\, ih(s)(x(s)) - \delta_\alpha(x)\| \to 0$, i.e., the graph limit of $\mathrm{ad}\, ih(s)|I$ is $\delta_\alpha|D(\delta_\alpha) \cap I$, which is equivalent to saying that $\max_{|t|\le 1} \|\mathrm{Ad}\, e^{ith(s)}(x) - \alpha_t(x)\| \to 0$ as $s \to \infty$ for $x \in I$. We will replace h by a function h' of $\mathbf{R}_+$ into I_{sa} such that $\|\mathrm{ad}\, h(s)(x(s)) - \mathrm{ad}\, ih'(s)(x(s))\| \to 0$.

For each $n \in \mathbf{N}$ let $M(n) = \max\{\|h(s)\| \;|0 \le s \le n\}$, which we may suppose is positive.

Since $D(\delta_\alpha) \cap I$ is a separable Banach $*$-algebra, let $(\mathcal{F}_n)$ be an increasing sequence of finite subsets of $D(\delta_\alpha) \cap I$ such that $\mathcal{F}_n^* = \mathcal{F}_n$ and $\bigcup_n \mathcal{F}_n$ is dense in $D(\delta_\alpha) \cap I$. Note that then $\bigcup_n \mathcal{F}_n$ is dense in I too.

Let $e_n \in I$ be such that $0 \le e_n \le 1$, $\|(1 - e_n)x(s)\| < (nM(n))^{-1}$, and $\|(1 - e_n)h(s)x(s)\| < n^{-1}$ for all $x \in \mathcal{F}_n$ and $s \in [0, n]$. We define a continuous function $e : \mathbf{R}_+ \to I_{sa}$ by

$$e(s) = (n - s)e_n + (s - n + 1)e_{n+1}, \quad s \in [n - 1, n],$$

where $n = 1, 2, \ldots$. Then if $s \in [n - 1, n]$, we get that

$$\|(1 - e(s))(x(s))\| < \frac{1}{nM(n)}$$

and

$$\|(1 - e(s))h(s)x(s)\| < \frac{1}{n}$$

for all $x \in \mathcal{F}_n$ and $s \in [0, n]$. Let $h'(s) = e(s)h(s)e(s) \in I$. Then, h' is a continuous function of $\mathbf{R}_+$ into I_{sa}. By computation, we get that $\|\mathrm{ad}\, ih'(s)(x(s)) - \mathrm{ad}\, ih(s)(x(s))\| < 4/n$ for $x \in \mathcal{F}_n$ and $s \in [0, n]$ because $\|[h'(s), x(s)] - [h(s), x(s)]\|$ is dominated by

$$\begin{aligned} &\|h(s)\|\|(e(s) - 1)x(s)\| + \|x(s)(e(s) - 1)\|\|h(s)\| \\ &+ \|(e(s) - 1)h(s)x(s))\| + \|x(s)h(s)(e(s) - 1)\|. \end{aligned}$$

Thus the graph limit of $\mathrm{ad}\, ih'(s)$ as $s \to \infty$ is $\delta_\alpha|D(\delta_\alpha) \cap I$. This concludes the proof.

For a flow α we define the *Connes spectrum* $\mathbf{R}(\alpha)$ as a closed subgroup of $\mathbf{R}$ (see [14] for details). In the following result we actually show that the flow α has the following property: For any non-empty open set $O \subset \mathbf{R}$ the spectral subspace $A^\alpha(O)$ has a central sequence (x_n) such that $\|x_n\| = 1$ and $\lim_{n\to\infty} \|x_n y\| = \|y\|,\ y \in A$, which insures that $\mathbf{R}(\alpha) = \mathbf{R}$.

Theorem 10. *Let A be a separable C^*-algebra. Then there is an asymptotically inner flow α on A such that the Connes spectrum $\mathbf{R}(\alpha)$ of α is full if and only if A is antiliminary.*

Proof. This is only a slight modification of Theorem 1.3 of [8].

Suppose that there is such a flow α on A. Let I be the largest ideal of A such that I is of type I and suppose that $I \neq 0$. Then, since α leaves an ideal of I invariant, it follows that $\alpha|I$ is universally weakly inner and that $\mathbf{R}(\alpha|I) = \{0\}$. Thus we get that $\mathbf{R}(\alpha) = \{0\}$ (since $\mathbf{R}(\alpha) \subset \mathbf{R}(\alpha|I)$), which is a contradiction. Hence I must be zero, i.e., A must be antiliminary.

Suppose that A is antiliminary. There are a countable family $\{\pi_i\}$ of irreducible representations of A such that $\bigcap_i \ker \pi_i = \{0\}$ and $\pi_i(A) \cap K(\mathcal{H}_{\pi_i}) = \{0\}$, where $K(\mathcal{H}_\pi)$ denotes the compact operators on the Hilbert space $\mathcal{H}_{\pi_i}$.

In the proof of 1.3 of [8] we worked with just one of such representations, say π, and constructed a bounded central sequence (h_n) in A_{sa} such that the flow α is defined as the limit of $\mathrm{Ad}\, e^{itH_n}$ and is covariant in π with the induced flow on $\pi(A)$ having the desired properties, where $H_n = h_1 + h_2 + \cdots + h_n$. (More precisely we also construct a bounded central sequence (b_n) such that various subsequences of (b_n) would produce a sequence (x_n) as stated before this theorem).

What we have to do now is to work with the direct sum $\pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_n$ at the n'th step of induction. The main tools we used in the proof are a version of Haagerup's result (Lemma 4.2 of [12]) and Kadison's transitivity, which are both available for finite direct sums of irreducible representations of the above type. Thus we can complete the proof in much the same way as in [8].

The flow α obtained this way is as a matter of fact asymptotically inner by Proposition 6; (h_n) satisfies that if $h : \mathbf{R}_+ \rightarrow A_{sa}$ is defined by $h(s) = H_{n-1} + (s - n + 1)h_n,\ s \in [n-1, n]$ with $H_0 = 0$ for $n = 1, 2, \ldots$, then $\mathrm{Ad}\, e^{ith(s)}$ converges to α_t.

Given an antiliminary C*-algebra we get the existence of a non-trivial asymptotically inner flow as above, but we do not know how many cocycle-conjugacy classes of such flows there are.

3 Lifting

Theorem 11. *Let A be a separable C^*-algebra and I a (closed) ideal of A. Let $B = A/I$ be the quotient of A by I with Q the canonical quotient map of A onto B. If β is an asymptotically inner flow on B, then there is an asymptotically inner flow α on A such that $Q\alpha = \beta Q$ and $\alpha|I$ is universally weakly inner.*

Proof. Let h be a continuous function of $\mathbf{R}_+$ into B_{sa} such that

$$\lim_{s \to \infty} \max_{|t| \leq 1} \|\beta_t(y) - \mathrm{Ad}\, e^{ith(s)}(y)\| \rightarrow 0$$

for every $y \in B$.

Let $(\mathcal{F}_n)$ be an increasing sequence of finite subsets of the unit ball A_1 of A such that the union $\bigcup_n \mathcal{F}_n$ is dense in A_1 and let (ϵ_n) be a decreasing sequence of positive numbers such that $\sum_n \epsilon_n < \infty$. We choose an increasing sequence (s_n) in $\mathbf{R}_+$ such that for any $s \geq s_n$ and $y \in Q(\mathcal{F}_n)$,

$$\max_{|t|\leq 1} \|\beta_t(y) - \mathrm{Ad}\, e^{ith(s)}(y)\| < \epsilon_n.$$

Since I is a separable ideal, there is an approximate identity (e_n) in I such that $e_n e_{n+1} = e_n$ and

$$\|[x, e_n]\| \to 0, \quad x \in A.$$

It also follows that $\|Q(x)\| = \lim_n \|x(1 - e_n)\|$ for any $x \in A$. We will use these facts in the arguments below.

We will find a continuous function H of $\mathbf{R}_+$ into A_{sa} such that $Q(H(s_n)) = h(s_n)$ and

$$\max_{|t|\leq 1} \|\mathrm{Ad}\, e^{itH(s_n)}(x) - \mathrm{Ad}\, e^{itH(s)}(x)\| < 7\epsilon_n$$

for $s \in [s_n, s_{n+1}]$ and $x \in \mathcal{F}_n$ and $H(s)e_n = H(s_n)e_n$ for $s \geq s_n$ and $n = 1, 2, \ldots$, where (e_n) is an approximate unit for I. Then it would follow that $\mathrm{Ad}\, e^{itH(s)}(x)$ converges uniformly in $t \in [-1, 1]$ for any $x \in A$ and thus defines an asymptotically inner flow α on A. Since $Q(H(s_n)) = h(s_n)$, we get that $Q\alpha_t = \beta_t Q$.

Since $\delta_\alpha|\overline{e_n A e_n} = \mathrm{ad}\, iH(s_n)|\overline{e_n A e_n}$, we get that $\delta_\alpha - \mathrm{ad}\, iH(s_n)$ vanishes on $\overline{e_n A e_n}$. Hence the flow $\alpha^{(-H(s_n))}$ generated by $\delta_\alpha - \mathrm{ad}\, iH(s_n)$ fixes each element of $\overline{e_n A e_n}$. This implies that if ϕ is a state of A such that $\|\phi|\overline{e_n A e_n}\| = 1$, then π_ϕ is covariant under α, which is just an inner perturbation of $\alpha^{(-H(s_n))}$. Since the set of states ϕ with the property $\|\phi|\overline{e_n A e_n}\| = 1$ for some n is dense in the states of I, we get that $\alpha|I$ is universally extendible (i.e., $t \mapsto \alpha_t^{**}(x)$ is weak*-continuous for $x \in I^{**}$), which is equivalent to being universally weakly inner [7]. Thus α has the desired properties.

Now we turn to the construction of $H : \mathbf{R}_+ \to A_{sa}$. We fix an approximate unit (e_n) for I such that $e_n e_{n+1} = e_n$ for all n. We choose an $H_1 \in A_{sa}$ such that $Q(H_1) = h(s_1)$ and set $H(s) = (s/s_1)H_1$ for $s \in [0, s_1]$.

Suppose that we have defined a continuous function $H : [0, s_n] \to A_{sa}$ such that

$$\max_{|t|\leq 1} \|\mathrm{Ad}\, e^{itH(s_k)}(x) - \mathrm{Ad}\, e^{itH(s)}(x)\| < 7\epsilon_k, \quad x \in \mathcal{F}_k$$

for $s \in [s_k, s_{k+1}]$ and $k = 1, 2, \ldots, n-1$ and

$$H(s)e_k = H(s_k)e_k, \quad s \in [s_k, s_{k+1}].$$

We find a continuous function $K : [s_n, s_{n+1}] \to A_{sa}$ such that $K(s_n) = H(s_n)$ and $Q(K(s)) = h(s),\ s \in [s_k, s_{k+1}]$. Since $\max_{|t|\leq 1} \|\mathrm{Ad}\, e^{ith(s_n)}(y) - \mathrm{Ad}\, e^{ith(s)}(y)\| < 2\epsilon_n$ for $s \in [s_n, s_{n+1}]$ and $y \in Q(\mathcal{F}_n)$, we get that

$$\|Q(\mathrm{Ad}\, e^{itK(s_n)}(x) - \mathrm{Ad}\, e^{itK(s)}(x))\| < 2\epsilon_n, \quad t \in [-1, 1],\ x \in \mathcal{F}_n$$

for $s \in [s_n, s_{n+1}]$. Hence there is an $e = e_m \in I$ for some $m \geq n$ such that

$$\|(1-e)(\mathrm{Ad}\, e^{itK(s)} \mathrm{Ad}\, e^{-itK(s_n)}(x) - x)\| < 2\epsilon_n, \quad t \in [-1,1],\ x \in \mathcal{F}_n$$

for $s \in [s_n, s_{n+1}]$.

Let γ denote the flow $t \mapsto \mathrm{Ad}\, e^{itK(s_n)} = \mathrm{Ad}\, e^{itH(s_n)}$ on A. Then $u^{(s)} : t \mapsto e^{itK(s)}e^{-itK(s_n)}$ is a γ-cocycle. We set $W(s) = K(s) - K(s_n)$ and note that

$$u_t^{(s)} = \sum_{n=0}^{\infty} \int \cdots \int_{\Gamma_n(t)} dt_1 dt_2 \cdots dt_n i^n \gamma_{t_1}(W(s))\gamma_{t_2}(W(s)) \cdots \gamma_{t_n}(W(s))$$

where $\Gamma_n(t)$ means $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq t$ if $t \geq 0$ and similar inequalities otherwise. Let $M = \max_{s \in [s_n, s_{n+1}]} \|W(s)\|$ and let $N \in \mathbf{N}$ be such that

$$\sum_{n=N+1}^{\infty} \frac{M^n}{n!} < \epsilon_n/8.$$

We will choose a $\delta > 0$ very small below. We find a finite sequence $(\sigma_i)_{i=0}^L$ in $[s_n, s_{n+1}]$ such that $s_n = \sigma_0 < \sigma_2 < \cdots < \sigma_L = s_{n+1}$ and $\|K(\sigma_i) - K(\sigma_{i-1})\| < \delta$ for $i = 1, 2, \ldots, L$. We find a finite sequence $(p_i)_{i=0}^K$ in I such that $0 \leq p_i \leq 1$, $p_0 e = e$, $p_i p_{i+1} = p_i$, and

$$\|[p_i, K(s)]\|,\ \|[p_i, x]\|, \quad s \in [s_n, s_{n+1}],\ x \in \mathcal{F}_n$$

are all very small.

For $i = 1, 2, \ldots, L$ we define

$$W_i = \sum_{k=1}^{i-1} (p_k - p_{k-1})^{1/2} W(\sigma_k)(p_k - p_{k-1})^{1/2} + (1-p_{i-1})^{1/2} W(\sigma_i)(1-p_{i-1})^{1/2}.$$

Since $\|W(\sigma_i) - W(\sigma_{i-1})\| < \delta$ and $(p_k - p_{k-1})^{1/2}$ and $W(\sigma)$ almost commute, it follows that $\|W_i - W_{i-1}\|$ is at most of the order of δ. It also follows that $W_i e_n = 0$ as $p_0 e_n = e_n$.

For $i = 1, 2, \ldots, L$ and $t \in \mathbf{R}$ we define $u_t^{(i)} = e^{it(K(s_n)+W_i)}e^{-itK(s_n)}$, which is a γ-cocycle. Since $\|W_i\| \leq M$, the norm difference of $u_t^{(i)}$ and

$$\sum_{n=0}^{N} \int \cdots \int_{\Gamma_n(t)} dt_1 \cdots dt_n i^n \gamma_{t_1}(W_i)\gamma_{t_2}(W_i) \cdots \gamma_{t_n}(W_i)$$

is smaller than $\epsilon_n/8$ for $t \in [-1, 1]$.

Let $j = 1, 2, \ldots, i-1$. We shall identify $(p_j - p_{j-1})u_t^{(i)}$. Since $\|(p_j - p_{j-1})(W_i - W(\sigma_j))\| < \delta$ and $(p_k - p_{k-1})^{1/2}$ is almost invariant under γ and are almost central as we have assumed, we can derive that

$$\|(p_j - p_{j-1})(\gamma_{t_1}(W_i) \cdots \gamma_{t_n}(W_i) - \gamma_{t_1}(W(\sigma_j)) \cdots \gamma_{t_n}(W(\sigma_j)))\|$$

is of the order of $\delta n M^{n-1}$, where we used that $|t_k| \leq 1$. Thus we get that

$$\|(p_j - p_{j-1})(u_t^{(i)} - e^{it(K(s_n)+W(\sigma_j))}e^{-itK(s_n)})\| < C\delta(e^M - 1) + \epsilon_n/4,$$

for $t \in [-1, 1]$, where C is a constant depending on N and M but can be made arbitrarily close to 1 by taking p_k's so that they almost commute with $K(s)$'s. So our choice of δ must be made such that $C\delta(e^M - 1)$ is smaller than $\epsilon_n/4$. In the same way we get that

$$\|(1 - p_{i-1})(u_t^{(i)} - e^{it(K(s_n)+W(\sigma_i))}e^{-itK(s_n)})\| < \epsilon_n/2$$

and

$$\|(1 - p_0)(u_t^{(i)} - 1)\| < \epsilon_n/2.$$

Since $1 - e$ dominates $p_j - p_{j-1}$ etc. and p_k's almost commute with $x \in \mathcal{F}_n$ and $K(s)$, we get that

$$\|(p_j - p_{j-1})(u_t^{(i)} x (u_t^{(i)})^* - x)\| < 2\epsilon_n + 2 \cdot \epsilon_n/2 = 3\epsilon_n,$$

for $t \in [-1, 1]$. Together with similar estimates we can show that

$$\|\mathrm{Ad}\, u_t^{(i)}(x) - x\| < 6\epsilon_n, \quad x \in \mathcal{F}_n, \quad t \in [-1, 1],$$

where we use that $(p_j - p_{j-1})(p_k - p_{k-1}) = 0$ for $|j - k| > 1$ etc. and that $[p_j - p_{j-1}, u_t^{(i)} x (u_t^{(i)})^* - x] \approx 0$ as closely as we wish.

Thus we have constructed $W_i,\ i = 1, 2, \cdots L$ such that $\|W_i - W_{i-1}\|$ is of the order of δ, $Q(W_i) = h(\sigma_i) - h(s_n)$, and

$$\|\mathrm{Ad}\, e^{it(K(s_n)+W_i)}(x) - \mathrm{Ad}\, e^{itK(s_n)}(x)\| < 6\epsilon_n, \quad x \in \mathcal{F}_n,\ t \in [-1, 1].$$

We define a continuous $H : [s_n, s_{n+1}] \rightarrow A_{sa}$ as follows: if $s \in [\sigma_{j-1}, \sigma_j]$, then

$$H(s) = \frac{\sigma_j - s}{\sigma_j - \sigma_{j-1}}(H(s_n) + W_{j-1}) + \frac{s - \sigma_{j-1}}{\sigma_j - \sigma_{j-1}}(H(s_n) + W_j).$$

Since $\|H(s) - (H(s_n) + W_j)\| < \delta$ for such s, we have that $\|e^{itH(s)} - e^{it(H(s_n)+W_j)}\| < \delta$ for $t \in [-1, 1]$. Thus we have that

$$\|\mathrm{Ad}\, e^{itH(s_n)}(x) - \mathrm{Ad}\, e^{itH(s)}(x)\| < 6\epsilon_n + 2\delta < 7\epsilon_n, \quad x \in \mathcal{F}_n, \quad t \in [-1, 1].$$

We also note that $H(s)e_n = H(s_n)e_n$ for $s \in [s_n, s_{n+1}]$ and that $Q(H(s_{n+1})) = Q(H(s_n) + W(s_{n+1})) = h(s_{n+1})$. This completes the proof.

Proposition 12. *Let A be a separable C^*-algebra and I an ideal of A. Let β be an asymptotically inner automorphism of the quotient $B = A/I$ in the sense that there is a continuous map $u : \mathbf{R}_+ \rightarrow \mathcal{U}(B)$ such that $\beta = \lim_{s\rightarrow\infty} \mathrm{Ad}\, u(s)$. Moreover suppose that u satisfies that $u(0) \in Q(\mathcal{U}(A))$, where Q is the quotient map of A onto B. Then there is an asymptotically inner automorphism α of A such that $Q\alpha = \beta Q$ and $\alpha|I$ is universally weakly inner.*

Proof. Let $(\mathcal{F}_n)$ be an increasing sequence of finite subsets of the unit ball A_1 such that $\bigcup_n \mathcal{F}_n$ is dense in A_1 and let (ϵ_n) be a decreasing sequence of positive numbers such that $\sum_n \epsilon_n < \infty$. Let $u : \mathbf{R}_+ \to \mathcal{U}(B)$ be as in the statement. We note that $\beta^{-1} = \lim_{s\to\infty} \mathrm{Ad}\, u(s)^*$ since $\|\beta(y) - \mathrm{Ad}\, u(s)(y)\| = \|\beta^{-1}\beta(y) - \mathrm{Ad}\, u(s)^*\beta(y)\|$.

We find an increasing sequence (s_n) in $\mathbf{R}_+$ such that for all $s \geq s_n$ and $y \in Q(\mathcal{F}_n)$,

$$\|\beta(y) - \mathrm{Ad}\, u(s)(y)\| < \epsilon_n,$$
$$\|\beta^{-1}(y) - \mathrm{Ad}\, u(s)^*(y)\| < \epsilon_n.$$

We will define a continuous map $U : \mathbf{R}_+ \to \mathcal{U}(A)$ such that $Q(U(s_n)) = u(s_n)$ for all n and

$$\|\mathrm{Ad}\, U(s)(x) - \mathrm{Ad}\, U(s_n)(x)\| < 5\epsilon_n,$$
$$\|\mathrm{Ad}\, U(s)^*(x) - \mathrm{Ad}\, U(s_n)^*(x)\| < 5\epsilon_n$$

for all $x \in \mathcal{F}_n$ and $s \in [s_n, s_{n+1}]$ and for all n.

Since $\sum_n \epsilon_n < \infty$, this implies that $(\mathrm{Ad}\, U(s)(x))$ is a Cauchy sequence for $x \in \bigcup_n \mathcal{F}_n$. Hence we can define an endomorphism α of A by $\alpha(x) = \lim_{s\to\infty} \mathrm{Ad}\, U(s)(x)$. Since we can also define an endomorphism γ of A by $\gamma(x) = \lim_{s\to\infty} \mathrm{Ad}\, U(s)^*(x)$ such that $\alpha\gamma = \mathrm{id} = \gamma\alpha$, we get that α is an automorphism. Since $Q(U(s_n)) = u(s_n)$, we also have that $Q\alpha = \lim_{n\to\infty} \mathrm{Ad}\, u(s_n)Q = \beta Q$.

We will require the map $U : \mathbf{R}_+ \to \mathcal{U}(A)$ to satisfy an additional condition as follows. There is an approximate unit (e_n) in I such that $e_n e_{n+1} = e_n$ and $U(s)U(s_n)^* e_n = e_n$ for $s \geq s_n$. This implies that $\alpha \mathrm{Ad}\, U(s_n)^* | I_n = \mathrm{id}|I_n$, where $I_n = \overline{e_n I e_n} = \overline{e_n A e_n}$. Hence if ϕ is a state of A such that $\|\phi|I_n\| = 1$, then $\phi\alpha\mathrm{Ad}\, U(s_n)^* = \phi$, i.e., π_ϕ is covariant under α. Since $\bigcup_n I_n$ is dense in I, we get that if ϕ is a state of I, then π_ϕ is covariant under α. Hence we can conclude that $\alpha|I$ is universally weakly inner [7].

Now we turn to the construction of such $U : \mathbf{R}_+ \to \mathcal{U}(A)$.

We have specified $u : \mathbf{R}_+ \to \mathcal{U}(B)$ and (s_n), such that

$$\|\mathrm{Ad}\, u(s)(y) - \mathrm{Ad}\, u(s_n)(y)\| < 2\epsilon_n,$$
$$\|\mathrm{Ad}\, u(s)^*(y) - \mathrm{Ad}\, u(s_n)^*(y)\| < 2\epsilon_n$$

for all $y \in Q(\mathcal{F}_n)$ and $s \in [s_n, s_{n+1}]$ and all $n = 1, 2, \ldots$. We also specify an approximate unit (e_n) for I such that $e_n e_{n+1} = e_n$ for all n.

Let $s_0 = 0$ and we choose a continuous map $U : [s_0, s_1] \to \mathcal{U}(A)$ such that $Q(U(s)) = u(s)$ for $s = s_0$ and $s = s_1$ (or for all $s \in [s_0, s_1]$). This is possible by the assumption.

Suppose that we have defined $U : [s_0, s_n] \to \mathcal{U}(A)$ as required, i.e., we have that $Q(U(s_k)) = u(s_k)$, $U(s)U(s_k)^* e_k = e_k$ for $s \in [s_k, s_{k+1}]$, and

$$\|\mathrm{Ad}\, U(s)(x) - \mathrm{Ad}\, U(s_n)(x)\| < 5\epsilon_k,$$
$$\|\mathrm{Ad}\, U(s)^*(x) - \mathrm{Ad}\, U(s_n)^*(x)\| < 5\epsilon_k$$

for all $x \in \mathcal{F}_k$ and $s \in [s_k, s_{k+1}]$ and for $k = 1, 2, \ldots, n-1$.

We choose a continuous $V : [s_n, s_{n+1}] \to \mathcal{U}(A)$ such that $V(s_n) = U(s_n)$ and $Q(V(s)) = u(s),\ s \in [s_n, s_{n+1}]$. Since

$$\begin{aligned} \|Q(\operatorname{Ad} V(s)(x) - \operatorname{Ad} V(s_n)(x))\| &< 2\epsilon_k, \\ \|Q(\operatorname{Ad} V(s)^*(x) - \operatorname{Ad} V(s_n)^*(x))\| &< 2\epsilon_k \end{aligned}$$

for $x \in \mathcal{F}_n$ and $s \in [s_n, s_{n+1}]$, we find an $e = e_m \in I$ for some $m \geq n$ such that

$$\begin{aligned} \|(1-e)(\operatorname{Ad} V(s)(x) - \operatorname{Ad} V(s_n)(x))\| &< 2\epsilon_k, \\ \|(1-e)(\operatorname{Ad} V(s)^*(x) - \operatorname{Ad} V(s_n)^*(x))\| &< 2\epsilon_k \end{aligned}$$

for $x \in \mathcal{F}_n$ and $s \in [s_n, s_{n+1}]$.

Let $\delta > 0$, which will be chosen later to be a sufficiently small constant. Let $(t_i)_{i=0}^N$ be a sequence in $[s_n, s_{n+1}]$ such that $s_n = t_0 < t_1 < t_2 < \cdots < t_N = s_{n+1}$ and

$$\|V(t_i) - V(t_{i-1})\| < \delta$$

for $i = 1, 2, \ldots, N$. We find a sequence $(f_i)_{i=0}^N$ in I such that $0 \leq f_i \leq 1$, $ef_0 = e$, $f_i f_{i+1} = f_i$, and

$$\|[f_i, V(t_j)]\| \approx 0, \quad \|[f_i, x]\| \approx 0$$

for all i, j and $x \in \mathcal{F}_n$. We define a sequence $(W_i)_{i=0}^N$ in $\tilde{A}$ by $W_0 = 1$ and

$$W_i = f_0 + \sum_{j=1}^{i-1} V(t_j)V(t_0)^*(f_j - f_{j-1}) + V(t_i)V(t_0)^*(1 - f_{i-1})$$

for $i = 1, 2, \ldots, N$. If $0 < j < i$, then we have that

$$\begin{aligned} W_i(f_j - f_{j-1}) &= V(t_{j-1})V(t_0)^*(f_{j-1} - f_{j-2})(f_j - f_{j-1}) \\ &\quad + V(t_j)V(t_0)^*(f_j - f_{j-1})^2 \\ &\quad + V(t_{j+1})V(t_0)^*(f_{j+1} - f_j)(f_j - f_{j-1}) \\ &= V(t_{j-1})V(t_0)^*(f_{j-1} - f_{j-1}^2) \\ &\quad + V(t_j)V(t_0)^*(f_j^2 + f_{j-1}^2 - 2f_{j-1}) \\ &\quad + V(t_{j+1})V(t_0)^*(f_j - f_j^2). \end{aligned}$$

By replacing $V(t_{j\pm 1})V(t_0)^*$ by $V(t_j)V(t_0)^*$, we get that

$$\|(W_i - V(t_j)V(t_0)^*)(f_j - f_{j-1})\| < \delta.$$

Moreover we have that $\|(W_i - 1)f_0\| < \delta$ and $\|(W_i - V(t_i)V(t_0)^*)(1 - f_{i-1})\| < \delta$. Assuming that $\|[V(t_j)V(t_0)^*, f_k - f_{k-1}]\| \approx 0$, we get that $\|(W_i^* W_i - 1)(f_j - f_{j-1})\| < 2\delta$ etc., where we have ignored an error of δ^2 (which may result if $\|W_i\| > 1$). Hence, by taking the summation over j and noting that $(f_j - f_{j-1})(f_k - f_{k-1}) = 0$ for $j > k+1$, we can conclude that

$$\|W_i^* W_i - 1\| < 4\delta.$$

In the same way we get that $\|W_i W_i^* - 1\| < 4\delta$, i.e., W_i is close to a unitary.

Note also that $\|W(t_i) - W(t_{i-1})\| < \delta$, which follows from

$$W_i - W_{i-1} = -V(t_{i-1})V(t_0)^*(1 - f_{i-1}) + V(t_i)V(t_0)^*(1 - f_{i-1}).$$

We will claim that $\Delta_i \equiv W_i U(s_n) x U(s_n)^* W_i^* - U(s_n) x U(s_n)^* \approx 0$ for $x \in \mathcal{F}_n$. Since $V(t_0) = U(s_n)$, this follows because if $0 < j < i$,

$$\Delta_i (f_j - f_{j-1}) \approx (V(t_j) x V(t_j)^* - V(t_0) x V(t_0)^*)(1 - e)(f_j - f_{j-1})$$

with an error of at most 2δ assuming that $\|[V(t_k), f_j - f_{j-1}]\| \approx 0$ and $\|[x, f_j - f_{j-1}]\| \approx 0$. Thus we get that

$$\|(W_i \mathrm{Ad}\, U(s_n)(x) W_i^* - \mathrm{Ad}\, U(s_n)(x))(f_j - f_{j-1})\| < 2\epsilon_n + 2\delta.$$

Together with similar inequalities with f_0 and $1 - f_{i-1}$ in place of $f_j - f_{j-1}$, we get that

$$\|W_i \mathrm{Ad}\, U(s_n)(x) W_i^* - \mathrm{Ad}\, U(s_n)(x)\| < 4(\epsilon_n + \delta).$$

Define a continuous function $W : [s_n, s_{n+1}] \rightarrow \mathcal{U}(A)$ by

$$W(t) = \frac{t_i - t}{t_i - t_{i-1}} W_{i-1} + \frac{t - t_{i-1}}{t_i - t_{i-1}} W_i, \quad t \in [t_{i-1}, t_i]$$

for $i = 1, 2, \ldots, N$. Since $\|W(t) - W_i\| < \delta$ for $t \in [t_{i-1}, t_i]$, we get that

$$\|W(t) \mathrm{Ad}\, U(s_n)(x) W(t)^* - \mathrm{Ad}\, U(s_n)(x)\| < 4\epsilon_n + 6\delta.$$

We let $U(s)$ be the unitary obtained from the polar decomposition of $W(s)U(s_n)$ for $s \in [s_n, s_{n+1}]$. Then by choosing $\delta > 0$ sufficiently small (or roughly $12\delta < \epsilon_n$; see below), we get that

$$\|\mathrm{Ad}\, U(s)(x) - \mathrm{Ad}\, U(s_n)(x)\| < 5\epsilon_n, \quad x \in \mathcal{F}_n.$$

Similarly we can also require that

$$\|\mathrm{Ad}\, U(s)^*(x) - \mathrm{Ad}\, U(s_n)^*(x)\| < 5\epsilon_n, \quad x \in \mathcal{F}_n.$$

By the construction we also have that $U(s)U(s_n)^* e_n = e_n$. This concludes the proof.

Lemma 13. *Let $\delta \in (0, 1/2)$ and let $W \in \tilde{A} = A + \mathbf{C}1$ be such that $\|WW^* - 1\| < \delta$ and $\|W^*W - 1\| < \delta$. If U denotes the unitary obtained from the polar decomposition of W, then $\|W\| < 2$ and $\|U - W\| < 2\delta$. Hence for $x \in A$ with $\|x\| \le 1$, it follows that $\|UxU^* - WxW^*\| < 6\delta$.*

The proof of this lemma is standard.

References

1. O. Bratteli and A. Kishimoto, Generation of semigroups and two-dimensional quantum lattice systems, J. Funct. Analysis 35 (1980), 344-368.
2. O. Bratteli and D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics, I, Springer, 1979.
3. O. Bratteli and D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics, II, Springer, 1981.
4. G.A. Elliott, Universally weakly inner one-parameter automorphism groups of separable C*-algebras, Math. Scand. 45 (1979), 139–146.
5. E. Kirchberg and N.C. Phillips, Embedding of exact C*-algebras in the Cuntz algebra $\mathcal{O}_2$, J. reine angew. Math. 525 (2000), 17–53.
6. E. Kirchberg and N.C. Phillips, Embedding of continuous fields of C*-algebras in the Cuntz algebra $\mathcal{O}_2$, J. reine angew. Math. 525 (2000), 55–94.
7. A. Kishimoto, Universally weakly inner one-parameter automorphism groups of C*-algebras, Yokohama Math. J. 30 (1982), 141–149.
8. A. Kishimoto, Approximately inner flows on separable C*-algebras, Rev. Math. Phys. 14 (2002), 649–673.
9. A. Kishimoto, Rohlin flows on the Cuntz algebra $\mathcal{O}_2$, Internat. J. Math. 13 (2002), 1065–1094.
10. A. Kishimoto, Rohlin property for flows, Contemporary Math. 335 (2003), 195–207.
11. A. Kishimoto, Central sequence algebras of a purely infinite simple C*-algebra, Canad. J. Math. 56 (2004), 1237–1258.
12. A. Kishimoto, N. Ozawa, and S. Sakai, Homogeneity of the pure state space of a separable C*-algebra, Canad. Math. Bull. 46 (2003), 365–372.
13. D. Olesen and G.K. Pedersen, Groups of automorphisms with spectrum condition and the lifting problem, Commun. Math. Phys. 51 (1976), 85–95.
14. G.K. Pedersen, C^*-algebras and their automorphism groups, Academic Press, 1979.
15. S. Sakai, Operator Algebras in Dynamical Systems, Cambridge Univ. Press, 1991.

Remarks on Free Entropy Dimension

Dimitri Shlyakhtenko *

Department of Mathematics, UCLA, Los Angeles, CA 90095
shlyakht@math.ucla.edu

Summary. We prove a technical result, showing that the existence of a closable unbounded dual system in the sense of Voiculescu is equivalent to the finiteness of free Fisher information. This approach allows one to give a purely operator-algebraic proof of the computation of the non-microstates free entropy dimension for generators of groups carried out in an earlier joint work with I. Mineyev [4]. The same technique also works for finite-dimensional algebras.

We also show that Voiculescu's question of semi-continuity of free entropy dimension, as stated, admits a counterexample. We state a modified version of the question, which avoids the counterexample, but answering which in the affirmative would still imply the non-isomorphism of free group factors.

Introduction

Free entropy dimension was introduced by Voiculescu [7, 8, 9] both in the context of his microstates and non-microstates free entropy. We refer the reader to the survey [11] for a list of properties as well as applications of this quantity in the theory of von Neumann algebras.

The purpose of this note is to discuss several technical aspects related to estimates for free entropy dimension.

The first deals with existence of "Dual Systems of operators", which were considered by Voiculescu [9] in connection with the properties of the difference quotient derivation, which is at the heart of the non-microstates definition of free entropy. We prove that if one considers dual systems of closed unbounded operators (as opposed to bounded operators as in [9]), then existence of a dual system becomes equivalent to finiteness of free Fisher information. Using these ideas allows one to give a purely operator-algebraic proof of the expression for the free entropy dimension of a set of generators of a group algebra in terms of the L^2 Betti numbers of the group [4], clarifying the reason for why the equality holds in the group case. We also point out that for the same

*Research supported by the National Science Foundation.

reason one is able to express the non-microstates free entropy dimension of an n-tuple of generators of a finite-dimensional von Neumann algebra in terms of its L^2 Betti numbers. In particular, the microstates and non-microstates free entropy dimension is the same in this case.

The second aspect deals with the question of semi-continuity of free entropy dimension, as formulated by Voiculescu in [7, 8]. We point out that a counterexample exists to the question of semi-continuity, as stated. However, the possibility that the free entropy dimension is independent of the choice of generators of a von Neumann algebra is not ruled out by the counterexample.

1 Unbounded Dual Systems and Derivations

1.1 Non-Commutative Difference Quotients and Dual Systems

Let $X_1, \dots, X_n$ be an n-tuple of self-adjoint elements in a tracial von Neumann algebra M. In [9], Voiculescu considered the densely defined derivations ∂_j defined on the polynomial algebra $\mathbb{C}(X_1, \dots, X_n)$ generated by $X_1, \dots, X_n$ and with values in $L^2(M) \otimes L^2(M) \cong HS(L^2(M))$, the space of Hilbert-Schmidt operators on $L^2(M)$. If we denote by $P_1 : L^2(M) \to L^2(M)$ the orthogonal projection onto the trace vector 1, then the derivations ∂_j are determined by the requirement that $\partial_j(X_i) = \delta_{ij} P_1$.

Voiculescu showed that if $\partial_j^*(P_1)$ exists, then ∂_j is closable. This is of interest because the existence of $\partial_j^*(P_1)$, $j = 1, \dots, n$ is equivalent to finiteness of the free Fisher information of $X_1, \dots, X_n$ [9].

Also in [9], Voiculescu introduced the notion of a "dual system" to $X_1, \dots, X_n$. In his definition, such a dual system consists of an n tuple of operators $Y_1, \dots, Y_n$, so that $[Y_i, X_j] = \delta_{ij} P_1$, where Although Voiculescu required that the operators Y_j be anti-self-adjoint, it will be more convenient to drop this requirement. However, this is not a big difference, since if $(Y_1', \dots, Y_n')$ is another dual system, then $[Y_i - Y_i', X_j] = 0$ for all i, j, and so $Y_i - Y_i'$ belongs to the commutant of $W^*(X_1, \dots, X_n)$.

Note that the existence of a dual system is equivalent to the requirement that the derivations $\partial_j : \mathbb{C}(X_1, \dots, X_n) \to HS \subset B(L^2(M))$ are inner as derivations into $B(L^2(M))$. In particular, Voiculescu showed that if a dual system exists, then $\partial_j : L^2(M) \to HS$ are actually closable, and $\partial_j^*(P_1)$ is given by $(Y_j - JY_j^*J)1$. However, the existence of a dual system is a stronger requirement than the existence of $\partial_j^*(P_1)$.

1.2 Dual Systems of Unbounded Operators

More generally, given an n-tuple $T = (T_1, \dots, T_n) \in HS^n$, we may consider a derivation $\partial_T : \mathbb{C}(X_1, \dots, X_n) \to HS$ determined by $\partial_T(X_j) = T_j$ [6]. The particular case of ∂_j corresponds to $T = (0, \dots, P_1, \dots, 0)$ (P_1 in j-th place).

Theorem 1. *Let $T \in HS^n$ and assume that $M = W^*(X_1, \ldots, X_n)$. The following are equivalent:*
(a) $\partial_T^*(P_1)$ exists;
(b) There exists a closable unbounded operator $Y : L^2(M) \to L^2(M)$, whose domain includes $\mathbb{C}(X_1, \ldots, X_n)$, so that $Y1 = 0$ and 1 belongs to the domain of Y^*, and so that $[Y, X_j] = T_j$.

Proof. Assume first that (b) holds. Let $\xi = (Y - JY^*J)1 = JY^*1$, which by assumptions on Y makes sense. Then for any polynomial $Q \in \mathbb{C}(X_1, \ldots, X_n)$,

$$\begin{aligned}\langle \xi, P\rangle &= \langle (Y - JY^*J)1, Q1\rangle \\ &= \langle [Y, Q]1, 1\rangle \\ &= Tr(P_1[Y, Q]) \\ &= \langle P_1, [Y, Q]\rangle_{HS} \\ &= \langle P_1, \partial_T(Q)\rangle_{HS},\end{aligned}$$

since the derivations $Q \mapsto \partial_T(Q)$ and $Q \mapsto [Y, Q]$ have the same values on generators and hence are equal on $\mathbb{C}(X_1, \ldots, X_n)$. But this means that $\xi = \partial_T^*(P_1)$.

Assume now that (a) holds. If we assume that $Y1 = 0$, then the equation $[Y, X_j] = T_j$ determines an operator $Y : \mathbb{C}(X_1, \ldots, X_n) \to L^2(M)$. Indeed, if Q is a polynomial in $X_1, \ldots, X_n$, then we have

$$Y(Q \cdot 1) = [Y, Q] \cdot 1 - Q(Y \cdot 1) = [Y, Q] \cdot 1 = \partial_T(Q) \cdot 1.$$

To show that the operator Y that we have thus defined is closable, it is sufficient to prove that a formal adjoint can be defined on a dense subset. We define Y^* on $Q \in \mathbb{C}(X_1, \ldots, X_n)$ by

$$Y^*(Q \cdot 1) = -(\partial_T(Q^*))^* \cdot 1 + \partial_T^*(P_1).$$

Hence $Y^* \cdot 1 = \partial_T^*(P_1)$ and Y^* satisfies $[Y^*, Q] = -(\partial_T(Q^*))^* = -[Y, Q^*]^*$.

It remains to check that $\langle YQ \cdot 1, R \cdot 1\rangle = \langle Q \cdot 1, Y^*R \cdot 1\rangle$, for all $Q, R \in \mathbb{C}(X_1, \ldots, X_n)$. We have:

$$\begin{aligned}\langle YQ \cdot 1, R \cdot 1\rangle &= \langle [Y, Q] \cdot 1, R \cdot 1\rangle \\ &= \langle 1, -[Y^*, Q^*]R \cdot 1\rangle \\ &= \langle 1, Q^*Y^*R \cdot 1\rangle - \langle 1, Y^*Q^*R \cdot 1\rangle \\ &= \langle Q \cdot 1, Y^*R \cdot 1\rangle - \langle 1, Y^*Q^*R \cdot 1\rangle.\end{aligned}$$

Hence it remains to prove that $\langle 1, Y^*Q^*R \cdot 1\rangle = 0$. To this end we write

$$\begin{aligned}\langle 1, Y^*Q^*R \cdot 1\rangle &= \langle 1, [Y^*, Q^*R] \cdot 1\rangle - \langle 1, Q^*RY^* \cdot 1\rangle \\ &= \langle [R^*Q, Y] \cdot 1, 1\rangle - \langle R^*Q \cdot 1, Y^* \cdot 1\rangle \\ &= Tr([R^*Q, Y]P_1) - \langle R^*Q \cdot 1, \partial_T^*(P_1)\rangle \\ &= \langle \partial_T(R^*Q), P_1\rangle_{HS} - \langle \partial_T(R^*Q), P_1\rangle_{HS} = 0.\end{aligned}$$

Thus Y is closable.

Corollary 2. *Let* $M = W^*(X_1, \ldots, X_n)$*. Then* $\Phi^*(X_1, \ldots, X_n) < +\infty$ *if and only if there exist unbounded essentially anti-symmetric operators* $Y_1, \ldots, Y_n : L^2(M) \to L^2(M)$ *whose domain includes* $\mathbb{C}(X_1, \ldots, X_n)$*, and which satisfy* $[Y_j, X_i] = \delta_{ji} P_1$*.*

Proof. A slight modification of the first part of the proof of Theorem 1 gives that if $Y_1, \ldots, Y_n$ exist, then $\partial_j^*(P_1) = (Y_j - JY_jJ)1$ and hence $\Phi^*(X_1, \ldots, X_n)$ (which is by definition $\sum_j \|\partial_j^*(P_1)\|_2^2$) is finite.

Conversely, if $\Phi^*(X_1, \ldots, X_n) < +\infty$, then $\partial_j^*(P_1)$ exists for all j. Hence by Theorem 1, we obtain non-self adjoint closable unbounded operators $Y_1, \ldots, Y_n$ so that the domains of $Y_j$and Y_j^* include $\mathbb{C}(X_1, \ldots, X_n)$, and so that $[Y_j, X_i] = \delta_{ji} P_1$. Now since $X_j = X_j^*$ we also have $[Y_j^*, X_i] = -\delta_{ji} P_1^* = -\delta_{ji} P_1$. Hence if we set $\tilde{Y}_j = \frac{1}{2}(Y_j - Y_j^*)$, we obtain the desired operators.

2 Dual Systems and L^2 Cohomology

Let as before $X_1, \ldots, X_n \in (M, \tau)$ be a family of self-adjoint elements.

In conjunctions with estimates on free entropy dimension [6, 4] and L^2 cohomology [2], it is interesting to consider the following spaces:

$$H_0 = \mathrm{cl}\,\{T = (T_1, \ldots, T_n) \in HS^n : \\ \exists Y \in B(L^2(M)) \;\; [Y, X_j] = T_j\}.$$

Here cl refers to closure in the Hilbert-Schmidt topology. We also consider

$$H_1 = \mathrm{span}\;\; \mathrm{cl}\,\{T = (T_1, \ldots, T_n) \in HS^n : \exists Y = Y^* \text{ unbounded densely defined} \\ \text{with 1 in the domain of } Y,\; [Y, X_j] = T_j,\; j = 1, \ldots, n\}.$$

Note that in particular, $H_0 \subset H_1$.

One has the following estimates [6, 2]:

$$\dim_{M \bar{\otimes} M^o} H_0 \leq \delta^*(X_1, \ldots, X_n) \leq \delta^\star(X_1, \ldots, X_n) \leq \Delta(X_1, \ldots, X_n).$$

The main result of this section is the following theorem, whose proof has similarities with the Sauvageot's theory of quantum Dirichlet forms [5]:

Theorem 3. $H_0 = H_1$*.*

Proof. It is sufficient to prove that H_0 is dense in H_1.

Let $T = (T_1, \ldots, T_n) \in HS^n$ be such that $T_j^* = T_j = [\mathrm{i}A, X_j]$, $j = 1, \ldots, n$, with $A = A^*$ a closed unbounded operator and 1 in the domain of A.

For each $0 < R < \infty$, let now $f_R : \mathbb{R} \to \mathbb{R}$ be a C^∞ function, so that

1. $f_R(x) = x$ for all $-R \leq x \leq R$;
2. $|f_R(x)| \leq R + 1$ for all x;
3. the difference quotient $g_R(s, t) = \frac{f_R(s) - f_R(t)}{s - t}$ is uniformly bounded by 2.

Let $A_R = f_R(A)$ and let $T_j^{(R)} = [\mathrm{i}A_R, X_j]$. Note that for each R, $T^{(R)} = (T_1^{(R)}, \dots, T_n^{(R)}) \in H_0$. Hence it is sufficient to prove that $T^{(R)} \to T$ in Hilbert-Schmidt norm as $R \to \infty$.

Let $\mathcal{A} \cong L^\infty(\mathbb{R}, \mu)$ be the von Neumann algebra generated by the spectral projections of A; hence $A_R \in \mathcal{A}$ for all R. If we regard $L^2(M)$ as a module over A, then $HS = L^2(M)\bar{\otimes}L^2(M)$ is a bimodule over $\mathcal{A}$, and hence a module over $\mathcal{A}\bar{\otimes}\mathcal{A} \cong L^\infty(\mathbb{R}^2, \mu \times \mu)$ in such a way that if s, t are coordinates on $\mathbb{R}^2$, and $Q \in HS$, then $sQ = AQ$ and $tQ = QA$ (more precisely, for any bounded measurable function f, $f(s)Q = f(A)Q$ and $f(t)Q = Qf(A)$). In particular, we can identify, up to multiplicity, HS with $L^2(\mathbb{R}^2, \mu \times \mu)$.

It is not hard to see that then

$$[f(A), X_j] = g \cdot [A, X_j],$$

where g is the difference quotient $g(s,t) = (f(s) - f(t))/(s - t)$. Indeed, it is sufficient to verify this equation on vectors in $\mathbb{C}[X_1, \dots, X_n]$ for f a polynomial in A, in which case the result reduces to

$$[A^n, X_j] = \sum_{k=0}^{n-1} A^k [A, X_j] A^{n-k-1} = \frac{s^n - t^n}{s - t} \cdot [A, X_j].$$

It follows that

$$T_j^{(R)} = [A_R, X_j] = [f_R(A), X_j] = g_R(A) \cdot [A, X_j] = g_R(A) \cdot T_j.$$

Now, since $g_R(A)$ are bounded and $g_R(A) = 1$ on the square $-R \le s, t \le R$, it follows that multiplication operators $g_R(A)$ converge to 1 ultra-strongly as $R \to \infty$. Since HS is a multiple of $L^2(\mathbb{R}^2, \mu \times \mu)$, it follows that $g_R(A)T_j \to T_j$ in Hilbert-Schmidt norm. Hence $T^{(R)} \to T$ as $R \to \infty$.

As a corollary, we re-derive the main result of [4] (the difference is that we use Theorem 3 instead of the more combinatorial argument [4]; we sketch the proof to emphasize the exact point at which the fact that we are dealing with a group algebra becomes completely clear):

Corollary 4. *Let $X_1, \dots, X_n$ be generators of the group algebra $\mathbb{C}\Gamma$. Then $\delta^\star(X_1, \dots, X_n) = \delta^*(X_1, \dots, X_n) = \Delta(X_1, \dots, X_n) = \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1$, where $\beta_j^{(2)}(\Gamma)$ are the L^2-Betti numbers of Γ.*

Proof. (Sketch). We first point out that in the preceding we could have worked with self-adjoint families $F = (X_1, \dots, X_n)$ rather than self-adjoint elements (all we ever needed was that $X \in F \Rightarrow X^* \in F$).

By [4], we may assume that $X_j \in \Gamma \subset \mathbb{C}\Gamma$, since the dimension of H_0 depends only on the pair $\mathbb{C}(X_1, \dots, X_n)$ and its trace.

Recall [2] that $\Delta(X_1, \dots, X_n) = \dim_{M\bar{\otimes}M^o} H_2$, where

$$H_2 = \{(T_1, \dots, T_n) \in HS : \exists Y^{(k)} \in HS \text{ s.t. } [Y^{(k)}, X_j] \to T_j \text{ weakly}\}.$$

By [2], $\Delta(X_1,\dots,X_n)=\beta_1^{(2)}(\Gamma)-\beta_0^{(2)}(\Gamma)+1$; moreover, from the proof we see that in the group case,

$$H_2=\mathrm{cl}\{MXM\},$$

where

$$X=\{(T_1X_1,\dots,T_nX_n):T_j\in\ell^2(\Gamma),\\ T_j \text{ is the value of some } \ell^2 \text{ group cocycle on } X_j\},$$

and where we think of $\ell^2(\Gamma)\subset HS$ as "diagonal operators" by sending a sequence $(a_\gamma)_{\gamma\in\Gamma}\in\ell^2(\Gamma)$ to the Hilbert-Schmidt operator $\sum a_\gamma P_\gamma$, where P_γ is the rank 1 projection onto the subspace spanned by the delta function supported on γ.

Let $\mathfrak{F}$ be the space of all functions on Γ. Since the group cohomology $H^1(\Gamma;\mathfrak{F}(\Gamma))$ is clearly trivial, it follows that if c is any ℓ^2-cocycle on Γ, then $c(X_j)=f(X_j)-f(e)$, for some $f\in\mathfrak{F}$. Hence

$$X=\{([f,X_1],\dots,[f,X_n]):f\in\mathfrak{F}\}\cap HS^n.$$

Since *every element of* $\mathfrak{F}$ *is automatically an essentially self-adjoint operator on* $\ell^2(\Gamma)$, whose domain includes $\mathbb{C}\Gamma$ we obtain that

$$MXM\subset H_1.$$

In particular, $H_2\subset H_1$. Hence

$$\dim_{M\bar{\otimes}M^o}H_2=\beta_1^{(2)}(\Gamma)-\beta_0^{(2)}(\Gamma)+1\le\dim_{M\bar{\otimes}M^o}H_1\le\dim_{M\bar{\otimes}M_0}H_2,$$

which forces $H_1=H_2$. Since $H_0=H_1$, we get that in the following equation

$$\dim_{M\bar{\otimes}M^o}H_0\le\delta^*\le\delta^\star\le\dim_{M\bar{\otimes}M^o}H_2=\beta_1^{(2)}(\Gamma)-\beta_0^{(1)}(\Gamma)+1$$

all inequalities are forced to be equalities, which gives the result.

Corollary 5. *Let* (M,τ) *be a finite-dimensional algebra, and let* $X_1,\dots,X_n$ *be any of its self-adjoint generators. Then* $\delta^*(X_1,\dots,X_n)=\delta^\star(X_1,\dots,X_n)=\Delta(X_1,\dots,X_n)=1-\beta_0(M,\tau)=\delta_0(X_1,\dots,X_n)$.

Proof. As in the proof of the last corollary, we have the inequalities

$$\dim_{M\bar{\otimes}M^o}H_0\le\delta^*\le\delta^\star\le\dim_{M\bar{\otimes}M^o}H_2,$$

where

$$H_2=\{(T_1,\dots,T_n)\in HS:\exists Y^{(k)}\in HS \text{ s.t. } [Y^{(k)},X_j]\to T_j \text{ weakly}\}.$$

Since $L^2(M)$ is finite-dimensional, there is no difference between weak and norm convergence; hence H_2 is in the (norm) closure of $\{(T_1,\dots,T_n):\exists Y\in$

HS s.t. $T_j = [Y, X_j]\} \subset H_0$; since H_0 is closed, we get that $H_0 = H_1$ and so all inequalities become equalities. Moreover,

$$\dim_{M\bar{\otimes}M^o} H_2 = \Delta(X_1, \ldots, X_n) = 1 - \beta_0(M, \tau)$$

(see [2]).

Comparing the values of $1 - \beta_0(M, \tau)$ with the computations in [3] gives $\delta_0(X_1, \ldots, X_n) = \delta^*(X_1, \ldots, X_n)$.

3 Some Remarks on Semi-Continuity of Free Dimension

In [7, 8], Voiculescu asked the question of whether the free dimension δ satisfies the following semi-continuity property. Let $X_j^{(k)}, X_j \in (M, \tau)$ be self-adjoint variables, $j = 1, \ldots, n$, $k = 1, 2, \ldots$, and assume that $X_j^{(k)} \to X_j$ strongly, $\sup_k \|X_j^{(k)}\| < \infty$. Then is it true that

$$\liminf_k \delta(X_1^{(k)}, \ldots, X_n^{(k)}) \geq \delta(X_1, \ldots, X_n)?$$

As shown in [7, 8], a positive answer to this question (or a number of related questions, where δ is replaced by some modification, such as δ_0, δ^*, etc.) implies non-isomorphism of free group factors. In the case of δ_0. a positive answer would imply that the value of δ_0 is independent of the choice of generators of a von Neumann algebra.

Although this question is very natural from the geometric standpoint, we give a counterexample, which shows that some additional assumptions on the sequence $X_j^{(k)}$ are necessary. Fortunately, the kinds of properties of δ that would be required to prove the non-isomorphism of free group factors are not ruled out by this counterexample (see Question 8).

We first need a lemma.

Lemma 6. *Let $X_1, \ldots, X_n$ be any generators of the group algebra of the free group $\mathbb{F}_k$. Then $\delta_0(X_1, \ldots, X_n) = \delta(X_1, \ldots, X_n) = \delta^*(X_1, \ldots, X_n) = \delta^\star(X_1, \ldots, X_n) = k$.*

Proof. Note that by [1] we always have

$$\delta_0(X_1, \ldots, X_n) \leq \delta(X_1, \ldots, X_n) \leq \delta^*(X_1, \ldots, X_n) \leq \delta^\star(X_1, \ldots, X_n);$$

furthermore, by [4], $\delta^\star(X_1, \ldots, X_n) = k$. Since δ_0 is an algebraic invariant [10], $\delta_0(X_1, \ldots, X_n) = \delta_0(U_1, \ldots, U_k)$, where $U_1, \ldots, U_k$ are the free group generators. Then by [8], $\delta_0(U_1, \ldots, U_k) = k$. This forces equalities throughout.

Example 7. Let u, v be two free generators of $\mathbb{F}_2$, and consider the map $\phi : \mathbb{F}_2 \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ given by $\phi(u) = \phi(v) = 1$. The kernel of this map is a subgroup Γ of $\mathbb{F}_2$, which is isomorphic to $\mathbb{F}_3$, having as free generators, e.g. u^2, v^2 and uv. Let $X_1^{(k)} = \mathrm{Re}\ u^2$, $X_2^{(k)} = \mathrm{Im}\ u^2$, $Y_1^{(k)} = \mathrm{Re}\ v^2$, $Y_2^{(k)} = \mathrm{Im}\ v^2$, $Z_1^{(k)} = \mathrm{Re}\ uv$, $Z_2^{(k)} = \mathrm{Im}\ uv$, $W_1^{(k)} = \frac{1}{k}\mathrm{Re}\ u$, $W_2^{(k)} = \frac{1}{k}\mathrm{Im}\ v$.

Thus if $X_1 = \mathrm{Re}\ u^2$, $X_2 = \mathrm{Im}\ u^2$, $Y_1 = \mathrm{Re}\ v^2$, $Y_2 = \mathrm{Im}\ v^2$, $Z_1 = \mathrm{Re}\ uv$, $Z_2 = \mathrm{Im}\ uv$, $W_1 = 0$, $W_2 = 0$, then $X_j^{(k)} \to X_j$, $Y_j^{(k)} \to Y_j$, $Z_j^{(k)} \to Z_j$ and $W_j^{(k)} \to W_j$ (in norm, hence strongly).

Note finally that for k finite, $X_1^{(k)}, X_2^{(k)}, Y_1^{(k)}, Y_2^{(k)}, Z_1^{(k)}, Z_2^{(k)}, W_1^{(k,)}, W_2^{(k)}$ generate the same algebra as $u^2, v^2, uv, \frac{1}{k}v$, which is the same as the algebra generated by u and v, i.e., the entire group algebra of $\mathbb{F}_2$. Hence $\delta(X_1^{(k)}, X_2^{(k)}, Y_1^{(k)}, Y_2^{(k)}, Z_1^{(k)}, Z_2^{(k)}, W_1^{(k,)}, W_2^{(k)}) = 2$ by Lemma 6. Hence

$$\liminf_k \delta(X_1^{(k)}, X_2^{(k)}, Y_1^{(k)}, Y_2^{(k)}, Z_1^{(k)}, Z_2^{(k)}, W_1^{(k,)}, W_2^{(k)}) = 2$$

On the other hand, $X_1, X_2, Y_1, Y_2, Z_1, Z_2, W_1, W_2$ generate the same algebra as $u^2, v^2, uv, 0$, i.e., the group algebra of $\Gamma \cong \mathbb{F}_3$. Hence

$$\delta(X_1, X_2, Y_1, Y_2, Z_1, Z_2, W_1, W_2) = 3,$$

which is the desired counterexample.

The same example (in view of Lemma 6) also works for δ_0, δ^* and $\delta^\star$.

The following two versions of the question are not ruled out by the counterexample. If either version were to have a positive answer, it would still be sufficient to prove non-isomorphism of free group factors:

Question 8. (a) Let $X_j^{(k)}, X_j \in (M, \tau)$ be self-adjoint variables, $j = 1, \ldots, n$, $k = 1, 2, \ldots$, and assume that $X_j^{(k)} \to X_j$ strongly, $\sup_k \|X_j^{(k)}\| < \infty$. Assume that $X_1, \ldots, X_n$ generate M and that for each k, $X_1^{(k)}, \ldots, X_n^{(k)}$ also generate M. Then is it true that

$$\liminf_k \delta(X_1^{(k)}, \ldots, X_n^{(k)}) \geq \delta(X_1, \ldots, X_n)?$$

(b) A weaker form of the question is the following. Let $X_j^{(k)}, X_j, Y_j \in (M, \tau)$ be self-adjoint variables, $j = 1, \ldots, n$, $k = 1, 2, \ldots$, and assume that $X_j^{(k)} \to X_j$ strongly, $\sup_k \|X_j^{(k)}\| < \infty$. Assume that $Y_1, \ldots, Y_m$ generate M. Then is it true that

$$\liminf_k \delta(X_1^{(k)}, \ldots, X_n^{(k)}, Y_1, \ldots, Y_m) \geq \delta(X_1, \ldots, X_n, Y_1, \ldots, Y_m)?$$

We point out that in the case of δ_0, these questions are actually equivalent to each other and to the statement that $\delta_0(Z_1, \ldots, Z_n)$ only depends on the von Neumann algebra generated by $Z_1, \ldots, Z_n$.

Indeed, it is clear that (a) implies (b).

On the other hand, if we assume that (b) holds, then we can choose $X_j^{(k)}$ to be polynomials in $Y_1, \ldots, Y_m$, so that $\delta_0(X_1^{(k)}, \ldots, X_n^{(k)}, Y_1, \ldots, Y_m) = \delta_0(Y_1, \ldots, Y_m)$ by [10]. Hence $\delta_0(Y_1, \ldots, Y_m) \geq \delta_0(X_1, \ldots, X_n, Y_1, \ldots, Y_m) \geq \delta_0(Y_1, \ldots, Y_m)$, where the first inequality is by (b) and the second inequality is proved in [8]. Hence if $W^*(X_1, \ldots, X_n) = W^*(Y_1, \ldots, Y_m)$, then one has $\delta_0(X_1, \ldots, X_n) = \delta_0(X_1, \ldots, X_n, Y_1, \ldots, Y_m) = \delta_0(Y_1, \ldots, Y_m)$. Hence (b) implies that δ_0 is the same on any generators of M.

Lastrly, if we assume that δ_0 is an invariant of the von Neumann algebra, then (a) clearly holds, since the value of $\delta_0(X_1^{(k)}, \ldots, X_n^{(k)})$ is then independent of k and is equal to $\delta_0(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$.

References

1. P. Biane, M. Capitaine, and A. Guionnet. Large deviation bounds for matrix Brownian motion. *Invent. Math.*, 152(2):433–459, 2003.
2. A. Connes and D. Shlyakhtenko. L^2-homology for von Neumann algberas. Preprint math.OA/0309343, to appear in J. Reine Angew. Math.
3. Kenley Jung. The free entropy dimension of hyperfinite von Neumann algebras. *Trans. Amer. Math. Soc.*, 355(12):5053–5089 (electronic), 2003.
4. I. Mineyev and D. Shlyakhtenko. Non-microstates free entropy dimension for groups. Preprint, math.OA/0312242, to appear in GAFA.
5. J.-L. Sauvageot. Quantum dirichlet forms, differential calculus, and semigroups. In *Quantum Probability and Applications V*, volume 1442 of *Lecture Notes in Mathematics*, pages 531–536. Springer-Verlag, 1990.
6. D. Shlyakhtenko. Some estimates for non-microstates free entropy dimension, with applications to q-semicircular families. *IMRN*, 51:2757–2772, 2004.
7. D.-V. Voiculescu. The analogues of entropy and of Fisher's information measure in free probability theory II. *Invent. Math.*, 118:411–440, 1994.
8. D.-V. Voiculescu. The analogues of entropy and of Fisher's information measure in free probability theory, III. *Geometric and Functional Analysis*, 6:172–199, 1996.
9. D.-V. Voiculescu. The analogues of entropy and of Fisher's information measure in free probabilility, V. *Invent. Math.*, 132:189–227, 1998.
10. D.-V. Voiculescu. A strengthened asymptotic freeness result for random matrices with applications to free entropy. *IMRN*, 1:41 – 64, 1998.
11. D.-V. Voiculescu. Free entropy. *Bull. London Math. Soc.*, 34(3):257–278, 2002.

Notes on Treeability and Costs for Discrete Groupoids in Operator Algebra Framework

Yoshimichi Ueda*

Graduate School of Mathematics, Kyushu University, Fukuoka, 810-8560, Japan
ueda@math.kyushu-u.ac.jp

1 Introduction

These notes discuss recent topics in orbit equivalence theory in operator algebra framework. Firstly, we provide an operator algebraic interpretation of discrete measurable groupoids in the course of giving a simple observation, which re-proves (and slightly generalizes) a result on treeability due to Adams and Spatzier [2, Theorem 1.8], by using operator algebra techniques. Secondly, we reconstruct Gaboriau's work [14] on costs of equivalence relations in operator algebra framework with avoiding any measure theoretic argument. It is done in the same sprit as of [23] for aiming to make Gaboriau's beautiful work much more accessible to operator algebraists (like us) who are not much familiar with ergodic theory. As simple byproducts, we clarify what kind of results in [14] can or cannot be generalized to the non-principal groupoid case, and observe that the cost of a countable discrete group with regarding it as a groupoid (i.e., a different quantity from Gaboriau's original one [14, p.43]) is nothing less than the smallest number of its generators in sharp contrast with the corresponding ℓ^2-Betti numbers, see Remark 12 (2). The methods given here may be useful for further discussing the attempts, due to Shlyakhtenko [29][30], of interpreting Gaboriau's work on costs by the idea of free entropy (dimension) due to Voiculescu.

We introduce the notational convention we will employ; for a von Neumann algebra N, the unitaries, the partial isometries and the projections in N are denoted by N^u, N^{pi} and N^p, respectively. The left and right support projections of $v \in N^{pi}$ are denoted by $l(v)$ and $r(v)$, respectively, i.e., $l(v) := vv^*$ and $r(v) := v^*v$. We also mention that only von Neumann algebras with separable preduals will be discussed throughout these notes.

We should thank Damien Gaboriau who earnestly explained us the core idea in his work, and also thank Tomohiro Hayashi for pointing out an

*Supported in part by Grant-in-Aid for Young Scientists (B) 14740118.

insufficient point in a preliminary version. The present notes were provided in part for the lectures we gave at University of Tokyo, in 2004, and we thank Yasuyuki Kawahigashi for his invitation and hospitality.

2 A Criterion for Treeability

Let $M \supseteq A$ be an inclusion of (not necessarily finite) von Neumann algebras with a faithful normal conditional expectation $E_A^M : M \to A$. Let $\mathbb{K}(M \supseteq A)$ be the C^*-algebra obtained as the operator norm $\| \cdot \|_\infty$-closure of Me_AM on $L^2(M)$ with the Jones projection e_A associated with E_A^M, and it is called the algebra of relative compact operators associated with the triple $M \supseteq A$, E_A^M. We use the notion of Relative Haagerup Property due to Boca [4]. (Quite recently, Popa used a slightly different formulation of Relative Haagerup Property in the type II_1 setting, see [24], but we employ Boca's in these notes.) The triple $M \supseteq A$, E_A^M is said to have Relative Haagerup Property if there is a net of A-bimodule (unital) normal completely positive maps $\Psi_\lambda : M \to M$, $\lambda \in \Lambda$, with $E_A^M \circ \Psi_\lambda = E_A^M$ for every $\lambda \in \Lambda$ such that for a fixed (and hence any) faithful state $\varphi \in M_*$ with $\varphi \circ E_A^M = \varphi$ one has

- $\lim_\lambda \|\Psi_\lambda(x) - x\|_\varphi = 0$ for every $x \in M$, or equivalently $\lim_\lambda \Psi_\lambda = \mathrm{id}_M$ pointwisely in σ-strong topology;
- $\widehat{\Psi}_\lambda \in \mathbb{K}(M \supseteq A)$,

where $\widehat{\Psi}_\lambda$ is the bounded operator on $L^2(M)$ defined by $\widehat{\Psi}_\lambda \Lambda_\varphi(x) := \Lambda_\varphi(\Psi_\lambda(x))$ for $x \in M$ with the canonical injection $\Lambda_\varphi : M \to L^2(M)$.

The next lemma can be proved in the essentially same way as in [7], where group von Neumann algebras are dealt with. Although the detailed proof is now available in [18, Proposition 3.5], we give its sketch for the reader's convenience, with focusing the "only if" part, which we will need later.

Lemma 1. *Assume that $M = A \rtimes_\alpha G$, i.e., M is the crossed-product of A by an action α of a countable discrete group G. Suppose that the action α has an invariant faithful state $\phi \in A_*$. Then, the inclusion $M \supseteq A$ with the canonical conditional expectation $E_A^M : M \to A$ has Relative Haagerup Property if and only if G has Haagerup Property* (*see* [17],[7]).

Proof. (Sketch) Let λ_g, $g \in G$, be the canonical generators of G in $M = A \rtimes_\alpha G$. The "if" part is the easier implication. In fact, if G has Haagerup Property, i.e., there is a net of positive definite functions ψ_λ with vanishing at infinity such that $\psi_\lambda(g) \to 1$ for every $g \in G$, then the required Ψ_λ can be constructed in such a way that $\Psi_\lambda\big(\sum_g a(g)\lambda_g\big) := \sum_{g\in G} \psi_\lambda(g) a(g)\lambda_g$ for every finite linear combination $\sum_{g\in G} a(g)\lambda_g \in A \rtimes_\alpha G$, see [17, Lemma 1.1]. The "only if" part is as follows. Define $\psi_\lambda(g) := \phi \circ E_A^M\big(\Psi_\lambda(\lambda_g)\lambda_g^*\big)$, $g \in G$, and clearly $\psi_\lambda(g) \to 1$ for every $g \in G$. Let $\varepsilon > 0$ be arbitrary small. One can choose a $T = \sum_{i=1}^n x_i e_A y_i \in Me_AM$ with $\big\|\widehat{\Psi}_\lambda - T\big\|_\infty \le \varepsilon/2$.

Then, for $g \in G$ one has $\left|\psi_\lambda(g)\right| \le \frac{\varepsilon}{2} + \sum_{i=1}^n \left\|x_i\right\|_\infty \left\|E_A^M(y_i\lambda_g)\right\|_\phi$. Since $\left\|y_i\right\|_{\phi\circ E_A^M}^2 = \sum_{h\in G}\left\|E_A^M\left(y_i\lambda_h^*\right)\right\|_\phi^2$ (where it is crucial that ϕ is invariant under α), one can choose a finite subset K of G in such a way that every $g \in G\setminus K$ satisfies that $\left\|E_A^M\big(y_i\lambda_g\big)\right\|_\phi \le \varepsilon/\big(2\sum_{i=1}^n \|x_i\|_\infty\big)$. Then, $|\psi_\lambda(g)| \le \varepsilon$ for every $g \in G\setminus K$.

In what follows, we further assume that A is commutative. Denote $\mathcal{G}(M \supseteq A) := \left\{v \in M : v^*v, vv^* \in A^p,\ vAv^* = Avv^*\right\}$ and call it the full (normalizing) groupoid of A in M. When A is a MASA in M and $\mathcal{G}(M \supseteq A)$ generates M as von Neumann algebra, we call A a Cartan subalgebra in M, see [12]. Let us introduce a von Neumann algebraic formulation of the set of one-sheeted sets in a countable discrete measurable groupoid.

Definition 2. *An E_A^M-groupoid is a subset $\mathcal{G}$ of $\mathcal{G}(M \supseteq A)$ equipped with the following properties:*

- $u, v \in \mathcal{G} \Longrightarrow uv \in \mathcal{G}$;
- $u \in \mathcal{G} \Longrightarrow u^* \in \mathcal{G}$;
- $u \in A^{pi} \Longrightarrow u \in \mathcal{G}$ *(and, in particular,* $u \in \mathcal{G}, p \in A^p \Longrightarrow pu, up \in \mathcal{G}$);
- *Let* $\{u_k\}_k$ *be a (possibly infinite) collection of elements in* $\mathcal{G}$. *If the support projections and the range projections respectively form mutually orthogonal families, then* $\sum_k u_k \in \mathcal{G}$ *in* σ*-strong* topology*;
- *Each* $u \in \mathcal{G}$ *has a (possibly zero)* $e \in A^p$ *such that* $e \le l(u)$ *and* $E_A^M(u) = eu$;
- *Each* $u \in \mathcal{G}$ *satisfies that* $E_A^M(uxu^*) = uE_A^M(x)u^*$ *for every* $x \in M$.

Such a projection e as in the fifth is uniquely determined as the modulus part of the polar decomposition of $E_A^M(u)$. The sixth automatically holds, either when E_A^M is the (unique) τ-conditional expectation with a faithful tracial state $\tau \in M_*$ or when A is a MASA in M. The next two lemmas are proved based on the same idea as for [23, Proposition 2.2].

Lemma 3. *Let* $\mathcal{U}$ *be an (at most countably infinite) collection of elements in* $\mathcal{G}$, *and* $w_0 := 1, w_1, \dots$ *be the words in* $\mathcal{U} \sqcup \mathcal{U}^*$ *of reduced form in the formal sense with regarding* $u^{-1} = u^*$ *for* $u \in \mathcal{U}$. *Suppose that* $\mathcal{G}'' = A \vee \mathcal{U}''$ *as von Neumann algebra. Then, each* $v \in \mathcal{G}$ *has a partition* $l(v) = \sum_k p_k$ *in* A^p *with* $p_k v = p_k E_A^M\left(vw_k^*\right) w_k$ *for every* k. *Furthermore, each coefficient* $E_A^M\left(vw_k^*\right)$ *falls in* A^{pi} *and* $v = \sum_k p_k E_A^M\left(vw_k^*\right) w_k$ *in* σ*-strong* topology.*

Proof. By the fifth requirement of E_A^M-groupoids one can find a (unique) $e_k \in A^p$ in such a way that $e_k \le l(v)$ and $E_A^M\left(vw_k^*\right) = e_k v w_k^*$, i.e., $e_k v = E_A^M\left(vw_k^*\right) w_k$. Set $e := \bigvee_k e_k$, and choose a faithful state $\varphi \in M_*$ with $\varphi \circ E_A^M = \varphi$. Then, we get $\big(v - ev | aw_k\big)_\varphi := \varphi\big((aw_k)^*(v - ev)\big) = 0$ for every $a \in A$ and every k, where the sixth requirement is used crucially. Since the aw_k's give a total subset in $\mathcal{G}''$ in σ-strong topology, we concludes that $v = ev$ so that $e = l(v)$. Since A is commutative, one can construct $p_0, p_1, \dots \in A^p$

in such a way that $p_k \le e_k$ and $\sum_k p_k = e$. Then, we have $p_k v = p_k e_k v = p_k E_A^M\,(vw_k^*)\,w_k$ for each k.

Lemma 4. *Let $\mathcal{G}_0 \subseteq \mathcal{G}$ be an E_A^M-groupoid with a faithful normal conditional expectation $E_{M_0}^M : M \to M_0 := \mathcal{G}_0''$ with $E_A^M \circ E_{M_0}^M = E_A^M$. Then, each $u \in \mathcal{G}$ has a (unique) $p \in A^p$ such that $p \le l(u)$ and $E_{M_0}^M(u) = pu$.*

Proof. By the same method as for the previous lemma together with standard exhaustion argument one can construct an (at most countably infinite) subset $\mathcal{W}$ of $\mathcal{G}_0$ that possesses the following properties: $E_A^M\,(w_1 w_2^*) = \delta_{w_1,w_2} l(w_1)$ for $w_1, w_2 \in \mathcal{W}$; each $u \in \mathcal{G}$ has an orthogonal family $\{e_w(u)\}_{w\in\mathcal{W}}$ such that $e_w(u) \le l(uw^*)$ $(\le l(u) \wedge r(w))$ and $e_w(u)u = E_A^M(uw^*)w$; if $u \in \mathcal{G}$ is chosen from $\mathcal{G}_0$, then $l(u) = \sum_{w\in\mathcal{W}} e_w(u)$. Choose a faithful state $\varphi \in M_*$ with $\varphi \circ E_A^M = \varphi$. Set $p := \sum_{w\in\mathcal{W}} e_w(u) \le l(u)$, and we have $\big(E_{M_0}^M(u) - pu\big|aw\big)_\varphi := \varphi\big((aw)^*(E_{M_0}^M(u) - pu)\big) = 0$ for every aw, $a \in A, w \in \mathcal{W}$. Hence, we get $E_{M_0}^M(u) = pu$. The uniqueness follows from that for the (right) polar decomposition of $E_{M_0}^M(u)$.

Let $\begin{array}{ccc} \mathcal{G}_{11} & \supset & \mathcal{G}_{12} \\ \cup & & \cup \\ \mathcal{G}_{21} & \supset & \mathcal{G}_{22} \end{array}$ be E_A^M-groupoids and write $M_{ij} := \mathcal{G}_{ij}''$. Assume that there are faithful normal conditional expectations $E_{ij} : M \to M_{ij}$ with $E_A^M \circ E_{ij} = E_A^M$. In this case, Lemma 4 enables us to see that the following three conditions are equivalent: $\begin{array}{ccc} M_{11} & \supset & M_{12} \\ \cup & & \cup \\ M_{21} & \supset & M_{22} \end{array}$ forms a commuting square; $M_{22} = M_{12} \cap M_{21}$; and $\mathcal{G}_{22} = \mathcal{G}_{12} \cap \mathcal{G}_{21}$. Moreover, one also observes, in the similar way as above, that if two E_A^M-groupoids inside a fixed $\mathcal{G}$ generate the same intermediate von Neumann algebra between $\mathcal{G}'' \supseteq A$, then they must coincide. If $A = \mathbf{C}1$, then the image $\pi(\mathcal{G})$ with the quotient map $\pi : M^u \to M^u/\mathbb{T}1$ is a countable discrete group. The full groupoid $\mathcal{G}(M \supseteq A)$ itself becomes an E_A^M-groupoid when A is a MASA in M thanks to Dye's lemma ([10, Lemma 6.1]; also see [6]), which asserts the same as in Lemma 4 for $\mathcal{G}(M \supseteq A)$ without any assumption, when A is a Cartan subalgebra in M. (The non-finite case needs a recently well-established result in [3].) Moreover, the set of one-sheeted sets in a countable discrete measurable groupoid canonically gives an E_A^M-groupoid, where $M \supseteq A$ with $E_A^M : M \to A$ are constructed by the so-called regular representation. See just after the next lemma for this fact. Let us introduce the notions of graphings and treeings due to Adams [1] (also see [14], [29, Proposition 7.5]) in operator algebra framework. We call such a collection $\mathcal{U}$ as in Lemma 3, i.e, $\mathcal{G}'' = A \vee \mathcal{U}''$, a graphing of $\mathcal{G}$. On the other hand, a collection $\mathcal{U}$ of elements in $\mathcal{G}(M \supseteq A)$ (*n.b.*, not assumed to be a graphing) is said to be a treeing if $E_A^M(w) = 0$ for all words w in $\mathcal{U} \sqcup \mathcal{U}^*$ of reduced form in the formal sense. This is equivalent to that $\mathcal{U}$ is a $*$-free family (or equivalently, $\{A \vee \{u\}''\}_{u\in\mathcal{U}}$ is a free family of von Neumann algebras) with respect to E_A^M in the sense of Voiculescu (see e.g. [32, §§3.8]) since every element in $\mathcal{G}(M \supseteq A)$ normalizes A. We say that $\mathcal{G}$ has a treeing $\mathcal{U}$ when $\mathcal{U}$ is a treeing and a graphing of $\mathcal{G}$, and also $\mathcal{G}$ is treeable if $\mathcal{G}$ has a treeing.

Lemma 5. (cf. [17], [4]) *If an E_A^M-groupoid $\mathcal{G}$ has a treeing $\mathcal{U}$, then the inclusion $M(\mathcal{G}) := \mathcal{G}'' \supseteq A$ with $E_A^M\big|_{M(\mathcal{G})} : M(\mathcal{G}) \to A$ must have Relative Haagerup Property.*

Proof. We may and do assume $M = M(\mathcal{G})$ for simplicity. We first assume that $\mathcal{U}$ is a finite collection. Since $\mathcal{U}$ is a treeing, each $u \in \mathcal{U}$ satisfies that $N_u := A \vee \{u\}''$ can be decomposed into

$$\text{(i)} \quad N_u = \sum_{|m| \le n_u}^{\oplus} u^m A \quad \text{or} \quad \text{(ii)} \quad N_u = \sum_{m \in \mathbb{Z}}^{\oplus} u^m A$$

in the Hilbert space $L^2(M)$ via Λ_φ with a faithful state $\varphi \in M_*$ with $\varphi \circ E_A^M = \varphi$. Here, u^{-m} means the adjoint u^{*m} as convention. By looking at this description, it is not so hard to confirm that each triple $N_u \supseteq A$ with $E_A^M\big|_{N_u}$ satisfies Relative Haagerup Property. Namely, one can construct a net $\Psi_u^{(\varepsilon)} : N_u \to N_u$ of completely positive maps in such a way that

- $E_A^M \circ \Psi_u^{(\varepsilon)} = E_A^M\big|_{N_u}$;
- $\Psi_u^{(\varepsilon)}$ converges to id_{N_u} pointwisely, in σ-strong topology, as $\varepsilon \searrow 0$;
- $\widehat{\Psi}_u^{(\varepsilon)}$ falls into $\mathbb{K}\left(N_u \supseteq A\right)$ on $L^2(N_u) = \overline{\Lambda_\varphi(N_u)}$;
- $T_u^{(\varepsilon)} := \widehat{\Psi}_u^{(\varepsilon)}\big|_{L^2(N_u)^\circ}$ satisfies $\left\|T_u^{(\varepsilon)}\right\|_\infty = \exp(-\varepsilon)$ with $L^2(N_u)^\circ := (1 - e_A)L^2(N_u)$.

The case (i) is easy, that is,

$$\Psi_u^{(\varepsilon)} := e^{-\varepsilon}\,\mathrm{id}_{N_u} + (1 - e^{-\varepsilon})E_A^M\big|_{N_u} = E_A^M\big|_{N_u} + e^{-\varepsilon}\left(\mathrm{id}_{N_u} - E_A^M\big|_{N_u}\right)$$

converges to id_{N_u} pointwidely, in σ-strong topology, and one has

$$\widehat{\Psi}_u^{(\varepsilon)} = e_A + e^{-\varepsilon}\left(\sum_{0 \lneqq |m| \le n_u} u^m e_A u^{-m}\right) \in N_u e_A N_u. \tag{1}$$

The case (ii) needs to modify the standard argument [17, Lemma 1.1]. By using the cyclic representation of $\mathbb{Z}$ induced by the positive definite function $m \mapsto e^{-\varepsilon|m|}$ one can construct a sequence $s_k \in \ell^\infty(\mathbb{Z})$ satisfying that $\sum_k |s_k(m)|^2 < +\infty$ for every $m \in \mathbb{Z}$ and moreover that $\sum_k s_k(m_1)\overline{s_k(m_2)} = e^{-\varepsilon(|m_1 - m_2|)}$ for every pair $m_1, m_2 \in \mathbb{Z}$. Set $S_k := \sum_{m \in \mathbb{Z}} s_k(m) u^m e_A u^{-m}$ (on $L^2(N_u)$), and then the desired completely positive maps can be given by

$$\Psi_u^{(\varepsilon)} : x \in N_u \mapsto \sum_k S_k x S_k^* \in B\left(L^2\left(N_u\right)\right).$$

(Note here that $u^m e_A u^{-m}$ is the projection from $L^2(N_u)$ onto $\overline{\Lambda_\varphi(u^m A)}$.) In fact, it is easy to see that $\Psi_u^{(\varepsilon)}(u^m a) = e^{-\varepsilon|m|}u^m a$ for $m \in \mathbb{Z}$, $a \in A$, which shows that the range of $\Psi_u^{(\varepsilon)}$ sits in N_u and that $\Psi_u^{(\varepsilon)}$ converges to id_{N_u} pointwidely, in σ-strong topology. Moreover, one has

$$\widehat{\Psi}_u^{(\varepsilon)} = \sum_{m\in\mathbb{Z}} e^{-\varepsilon|m|}u^m e_A u^{-m} = \lim_{n\to\infty}\sum_{|m|\le n} e^{-\varepsilon|m|}u^m e_A u^{-m} \tag{2}$$

in operator norm.

Since $\mathcal{U}$ is a treeing, we have

$$\left(M, E_A^M\right) = \bigstar_{A \atop u\in\mathcal{U}}\left(N_u, E_A^M\big|_{N_u}\right).$$

Therefore, [4, Proposition 3.9] shows that the inclusion $M \supseteq A$ with E_A^M satisfies Relative Haagerup Property since we have shown that so does each $N_u \supseteq A$ with $E_A^M\big|_{N_u}$. However, we would like to give the detailed argument on this point for the reader's convenience. Thanks to $E_A^M \circ \Psi_u^{(\varepsilon)} = E_A^M\big|_{N_u}$, we can construct the free products of completely positive maps $\Psi^{(\varepsilon)} := \bigstar_{A \atop u\in\mathcal{U}}\Psi_u^{(\varepsilon)} : M \to M$, which is uniquely determined by the following properties:

- $E_A^M \circ \Psi^{(\varepsilon)} = E_A^M$;
- $\Psi^{(\varepsilon)}(x_1 x_2 \cdots x_\ell) = \Psi_{u_1}^{(\varepsilon)}(x_1)\,\Psi_{u_2}^{(\varepsilon)}(x_2)\cdots\Psi_{u_\ell}^{(\varepsilon)}(x_\ell)$ for $x_j^\circ \in \mathrm{Ker}E_A^M \cap N_{u_j}$ with $u_1 \neq u_2 \neq \cdots \neq u_\ell$.

(See [5, Theorem 3.8] in the most generic form at present.) Since each $\Psi_u^{(\varepsilon)}$ converges to id_{N_u} pointwidely in σ-strong topology, as $\varepsilon \searrow 0$, the above two properties enable us to confirm that so does $\Psi^{(\varepsilon)}$ to id_M. It is standard to see that

$$\widehat{\Psi}^{(\varepsilon)} = 1_{L^2(A)} \oplus \sum_{\ell\ge 1}^{\oplus}\ \sum_{u_1\neq u_2\neq\cdots\neq u_\ell}^{\oplus} T_{u_1}^{(\varepsilon)} \otimes_\varphi T_{u_2}^{(\varepsilon)} \otimes_\varphi \cdots \otimes_\varphi T_{u_\ell}^{(\varepsilon)}$$

in the free product representation

$$L^2(M) = L^2(A) \oplus \sum_{\ell\ge 1}^{\oplus}\ \sum_{u_1\neq u_2\neq\cdots\neq u_\ell}^{\oplus} L^2\left(N_{u_1}\right)^\circ \otimes_\varphi \cdots \otimes_\varphi L^2\left(N_{u_\ell}\right)^\circ \tag{3}$$

with $L^2(A) = \overline{\Lambda_\varphi(A)} \subseteq L^2(M)$, where $\otimes_\varphi$ means the relative tensor product operation over A with respect to $\varphi|_A \in A_*$ (see [27]). Notice that, with $x_j^\circ \in \mathrm{Ker}E_A^M \cap N_{u_j}$, $u_1 \neq u_2 \neq \cdots \neq u_\ell$,

$$\Lambda_\varphi\left(x_1^\circ x_2^\circ \cdots x_\ell^\circ\right) = \Lambda_\varphi\left(x_1^\circ\right) \otimes_\varphi \Lambda_\varphi\left(x_2^\circ\right) \otimes_\varphi \cdots \otimes_\varphi \Lambda_\varphi\left(x_\ell^\circ\right)$$

in (3), and hence by (1),(2), we have, via (3),

$$\widehat{\Psi}^{(\varepsilon)}\big|_{L^2\left(N_{u_1}\right)^\circ \otimes_\varphi L^2\left(N_{u_2}\right)^\circ \otimes_\varphi \cdots \otimes_\varphi L^2\left(N_{u_\ell}\right)^\circ}$$

$$= T_{u_1}^{(\varepsilon)} \otimes_\varphi T_{u_2}^{(\varepsilon)} \otimes_\varphi \cdots \otimes_\varphi T_{u_\ell}^{(\varepsilon)}$$
$$= \sum_{m_1, m_2, \dots, m_\ell} e^{-\varepsilon n}\, u_1^{m_1} u_2^{m_2} \cdots u_\ell^{m_\ell} e_A u_\ell^{-m_\ell} \cdots u_2^{-m_2} u_1^{-m_1}$$

with certain natural numbers $n = n(u_1, u_2, \dots, u_\ell; m_1, m_2, \dots, m_\ell)$ that converges to $+\infty$ as $|m_1|, |m_2| \dots, |m_\ell| \to \infty$ (as long as when it is possible to do so). Note also that

$$\left\| T_{u_1}^{(\varepsilon)} \otimes_\varphi T_{u_2}^{(\varepsilon)} \otimes_\varphi \cdots \otimes_\varphi T_{u_\ell}^{(\varepsilon)} \right\|_\infty \leq \left\| T_{u_1}^{(\varepsilon)} \right\|_\infty \cdot \left\| T_{u_2}^{(\varepsilon)} \right\|_\infty \cdots \left\| T_{u_\ell}^{(\varepsilon)} \right\|_\infty$$
$$= e^{-\ell\varepsilon} \longrightarrow 0 \quad (\text{as } \ell \to \infty).$$

By these facts, $\widehat{\Psi}^{(\varepsilon)}$ clearly falls in the operator norm closure of Me_AM since $\mathcal{U}$ is a finite collection. Hence, the net $\Psi^{(\varepsilon)}$ of completely positive maps on M provides a desired one showing that the inclusion $M \supseteq A$ with the E_A^M has Relative Haagerup Property.

Next, we deal with the case that $\mathcal{U}$ is an infinite collection. In this case, one should at first choose a filtration $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \cdots \nearrow \mathcal{U} = \bigcup_k \mathcal{U}_k$ by finite sub-collections. Then, instead of the above $\Psi^{(\varepsilon)}$ we consider the completely positive maps

$$\Psi_k^{(\varepsilon)} := \left(\bigstar_{A\, u \in \mathcal{U}_k} \Psi_u^{(\varepsilon)} \right) \circ E_{M_k}^M : M \to M_k := \bigvee_{u \in \mathcal{U}_k} N_u \left(= \bigstar_{A\, u \in \mathcal{U}_k} N_u \right) \to M_k \subseteq M$$

with the $\varphi \circ E_A^M$-conditional expectations $E_{M_k}^M : M \to M_k$. Since $M_1 \subseteq M_2 \subseteq \cdots \nearrow M = \bigvee_k M_k$, the non-commutative Martingale convergence theorem [8, Lemma 2] says that $E_{M_k}^M$ converges to id_M pointwidely, in σ-strong topology, as $k \to \infty$, and so does $\Psi_k^{(\varepsilon)}$ to id_M too, as $\varepsilon \searrow 0$, $k \to \infty$. We easily see that

$$\widehat{\Psi}_k^{(\varepsilon)} = 1_{L^2(A)} \oplus \sum_{\ell \geq 1}^{\oplus} \sum_{u_1 \neq u_2 \neq \cdots \neq \frac{u_\ell}{u_j} \in \mathcal{U}_k}^{\oplus} T_{u_1}^{(\varepsilon)} \otimes_\varphi T_{u_2}^{(\varepsilon)} \otimes_\varphi \cdots \otimes_\varphi T_{u_\ell}^{(\varepsilon)} \tag{4}$$

in (3). Note that the summation of each ℓth direct summand of (4) is taken over the alternating words in the fixed finite collection $\mathcal{U}_k$ of length ℓ, and thus the previous argument works for showing that $\widehat{\Psi}_k^{(\varepsilon)}$ falls into $\mathbb{K}(M \supseteq A)$. Hence, we are done.

Here, we briefly summarize some basic facts on von Neumann algebras associated with countable discrete measurable groupoids, see e.g. [16],[25]. Let Γ be a countable discrete measurable groupoid with unit space X, where X is a standard Borel space with a regular Borel measure. With a non-singular measure on X under Γ one can construct, in a canonical way, a pair $M(\Gamma) \supseteq A(\Gamma)$ of von Neumann algebra and distinguished commutative von Neumann subalgebra with $A(\Gamma) = L^\infty(X)$ and a faithful normal conditional expectation $E_\Gamma : M(\Gamma) \to A(\Gamma)$, by the so-called regular representation of Γ due to Hahn [16] (also see [25, Chap. II]), which generalizes Feldman-Moore's construction [12] for countable discrete measurable equivalence relations. Denote by $\mathcal{G}_\Gamma$ of Γ the set of "one-sheeted sets in Γ" or called "Γ-sets", i.e., measurable subsets of Γ, on which the mappings $\gamma \in \Gamma \mapsto \gamma\gamma^{-1}, \gamma^{-1}\gamma \in X$ are both injective. Note that $\mathcal{G}_\Gamma$ becomes an inverse semigroup with product $E_1E_2 := \{\gamma_1\gamma_2 : \gamma_1 \in E_1, \gamma_2 \in E_2, \gamma_1^{-1}\gamma_1 = \gamma_2\gamma_2^{-1}\}$ and inverse $E \mapsto E^{-1} := \{\gamma^{-1} : \gamma \in E\}$. Each $E \in \mathcal{G}_\Gamma$ gives an element $u(E) \in \mathcal{G}(M(\Gamma) \supseteq A(\Gamma))$ with the properties: Its left and right support projections $l(u(E)), r(u(E))$ coincide with the characteristic functions on $EE^{-1} = \{\gamma\gamma^{-1} : \gamma \in E\}, E^{-1}E = \{\gamma^{-1}\gamma : \gamma \in E\}$, respectively, in $L^\infty(X)$; The mapping $u : E \in \mathcal{G}_\Gamma \mapsto u(E) \in \mathcal{G}(M(\Gamma) \supseteq A(\Gamma))$ is an inverse semigroup homomorphism (being injective modulo null sets), where $\mathcal{G}(M(\Gamma) \supseteq A(\Gamma))$ is equipped with the inverse operation $u \mapsto u^*$; $E_\Gamma(u(E)) = eu(E)$ with the projection e given by the characteristic function on $X \cap E$; $E_\Gamma\big(u(E)xu(E)^*\big) = u(E)E_\Gamma(x)u(E)^*$ for every $x \in M(\Gamma)$. It is not difficult to see that $\mathcal{G}(\Gamma) := A(\Gamma)^{pi}u(\mathcal{G}_\Gamma) = \big\{au(E) \in \mathcal{G}(M(\Gamma) \supseteq A(\Gamma)) : a \in A(\Gamma)^{pi}, E \in \mathcal{G}_\Gamma\big\}$ is an E_Γ-groupoid, which generates $M(\Gamma)$ as von Neumann algebra. An (at most countably infinite) collection $\mathcal{E}$ of elements in $\mathcal{G}_\Gamma$ is called a graphing of Γ if it generates Γ as groupoid, or equivalently the smallest groupoid that contains $\mathcal{E}$ becomes Γ. If no word in $\mathcal{E} \sqcup \mathcal{E}^{-1}$ of reduced form in the formal sense intersects with the unit space X of strictly positive measure, then we call $\mathcal{E}$ a treeing of Γ. Then, it is not hard to see the following two facts: (i) the collection $u(\mathcal{E})$ of $u(E) \in \mathcal{G}(M(\Gamma) \supseteq A(\Gamma))$ with $E \in \mathcal{E}$ is a graphing of $\mathcal{G}(\Gamma)$ if and only if $\mathcal{E}$ is a graphing of Γ; and similarly, (ii) the collection $u(\mathcal{E})$ is a treeing of $\mathcal{G}(\Gamma)$ if and only if $\mathcal{E}$ is a treeing of Γ. With these considerations, the previous two lemmas immediately imply the following criterion for treeability:

Proposition 6. *Relative Haagerup Property of $M(\Gamma) \supseteq A(\Gamma)$ with E_Γ is necessary for treeability of countable discrete measurable groupoid Γ. In particular, any countably infinite discrete group without Haagerup Property has no treeable free action with finite invariant measure.*

Note that this follows from a much deeper result due to Hjorth (see [20, §28]) with the aid of Lemma 1 if a given Γ is principal or an equivalence relation. The above proposition clearly implies the following result of Adams and Spatzier:

Corollary 7. ([2, Theorem 1.8]) *Any countably infinite discrete group of Property T admits no treeable free ergodic action with finite invariant measure.*

Remark 8. Note that the finite measure preserving assumption is very important in the above assertions. In fact, any countably infinite discrete group of Property T has an amenable free ergodic action without invariant finite measure (e.g. the boundary actions of some word-hyperbolic groups and the translation actions of discrete groups on themselves).

3 Operator Algebra Approach to Gaboriau's Results

We explain how to re-prove Gaboriau's results [14] on costs of equivalence relations (and slightly generalize them to the groupoid setting) in operator algebra framework, avoiding any measure theoretic argument. Throughout this section, we keep and employ the terminologies in the previous section.

Let $\mathcal{E}$ be a graphing of a countable discrete measurable groupoid Γ with a non-singular probability measure μ on the unit space X. Following Levitt [22] and Gaboriau [14] the μ-cost of $\mathcal{E}$ is defined to be

$$C_\mu(\mathcal{E}) := \sum_{E\in\mathcal{E}} \frac{\mu\left(EE^{-1}\right)+\mu\left(E^{-1}E\right)}{2},$$

and the μ-cost of Γ by taking the infimum all over the graphings, that is,

$$C_\mu(\Gamma) := \inf\left\{C_\mu(\mathcal{E}) : \mathcal{E} \text{ graphing of } \Gamma\right\}.$$

In fact, if Γ is a principal one (or equivalently a countable discrete equivalence relation) with an invariant probability measure μ, the μ-cost of graphings and that of Γ coincide with Levitt and Gaboriau's ones.

Let $M \supseteq A$ be a von Neumann algebra and a distinguished commutative von Neumann subalgebra with a faithful normal conditional expectation $E_A^M : M \to A$, and $\mathcal{G}$ be an E_A^M-groupoid. For a faithful state $\varphi \in M_*$ with $\varphi\circ E_A^M = \varphi$, the φ-cost of a graphing $\mathcal{U}$ of $\mathcal{G}$ is defined to be

$$C_\varphi(\mathcal{U}) := \sum_{u\in\mathcal{U}} \frac{\varphi\left(l(u)+r(u)\right)}{2},$$

and that of $\mathcal{G}$ by taking the infimum all over the graphings of $\mathcal{G}$, that is,

$$C_\varphi(\mathcal{G}) := \inf\left\{C_\varphi(\mathcal{U}) : \mathcal{U} \text{ graphing of } \mathcal{G}\right\}.$$

We sometimes consider those cost functions C_φ for both graphings and E_A^M-groupoids with the same equations even when φ is not a state (but still normal and positive). When $\mathcal{G} = \mathcal{G}(\Gamma)$, i.e., the canonical E_Γ-groupoid associated with a countable discrete measurable groupoid Γ, it is plain to verify that $C_\varphi(\mathcal{G}(\Gamma)) = C_\mu(\Gamma)$ with the state $\varphi \in M(\Gamma)_*$ defined to be $\left(\int_X \cdot\ \mu(dx)\right)\circ E_\Gamma$. Therefore, it suffices to consider E_A^M-groupoids and their φ-costs to re-prove Gaboriau's results in operator algebra framework with generalizing it to the (even not necessary principal) groupoid setting, and indeed many of results in [14] can be proved purely in the framework. For example, we can show the following additivity formula of costs of E_A^M-groupoids:

Theorem 9. (cf. [14, Théoème IV.15]) *Assume that M has a faithful tracial state $\tau \in M_*$ with $\tau \circ E_A^M = \tau$. Let $\mathcal{G}_1 \supseteq \mathcal{G}_3 \subseteq \mathcal{G}_2$ be E_A^M-groupoids. Set $N_1 := \mathcal{G}_1''$, $N_2 := \mathcal{G}_2''$ and $N_3 := \mathcal{G}_3''$ (all of which clearly contains A), and let*

$E^M_{N_3} : M \to N_3$ *be the* τ*-conditional expectation (hence* $E^M_A \circ E^M_{N_3} = E^M_A$*). Suppose that*

$$\big(M, E^M_{N_3}\big) = \big(N_1, E^M_{N_3}|_{N_1}\big) \mathop{\bigstar}_{N_3} \big(N_2, E^M_{N_3}|_{N_2}\big),$$

or equivalently $\mathcal{G}_1$, $\mathcal{G}_2$ *are* $*$*-free with amalgamation* N_3 *with respect to* $E^M_{N_3}$*, and further that* A *is a MASA in* N_3 *so that* $\mathcal{G}_3 = \mathcal{G}\,(N_3 \supseteq A)$ *holds automatically, see the discussion just below Lemma 4. (Remark here that* A *needs not to be a MASA in* N_1 *nor* N_2*.) Then, if* N_3 *is hyperfinite, then the smallest* E^M_A*-groupoid* $\mathcal{G} = \mathcal{G}_1 \vee \mathcal{G}_2$ *that contains* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *satisfies that*

$$C_\tau(\mathcal{G}) = C_\tau(\mathcal{G}_1) + C_\tau(\mathcal{G}_2) - C_\tau(\mathcal{G}_3)$$

as long as when $C_\tau(\mathcal{G}_1)$ *and* $C_\tau(\mathcal{G}_2)$ *are both finite.*

This can be regarded as a slight generalization of one of the main results in [14] to the groupoid setting. In fact, let Γ be a countable discrete measurable groupoid with an invariant probability measure μ, and assume that it is generated by two countable discrete measurable subgroupoids Γ_1, Γ_2. If no alternating word in $\Gamma_1 \setminus \Gamma_3, \Gamma_2 \setminus \Gamma_3$ with $\Gamma_3 := \Gamma_1 \cap \Gamma_2$ intersects with the unit space of strictly positive measure, i.e., Γ is the "free product with amalgamation $\Gamma_1 \bigstar_{\Gamma_3} \Gamma_2$" (modulo null set), and Γ_3 is principal and hyperfinite, then the above formula immediately implies the formula $C_\mu(\Gamma) = C_\mu(\Gamma_1) + C_\mu(\Gamma_2) - C_\mu(\Gamma_3)$ as long as when $C_\mu(\Gamma_1)$ and $C_\mu(\Gamma_2)$ are both finite. Here, we need the same task as in [21].

Proving the above theorem needs several lemmas and propositions, many of which can be proved based on the essentially same ideas as in [14] even in operator algebra framework so that some of their details will be just sketched.

The next simple fact is probably known but we could not find a suitable reference.

Lemma 10. *Let* $\mathcal{G}$ *be an* E^M_A*-groupoid with* $M = \mathcal{G}''$*, and assume that* M *is finite. Then, if* $e, f \in A^p$ *are equivalent in* M*, denoted by* $e \sim_M f$*, in the sense of Murray-von Neumann (i.e.,* $l(u) = e$ *and* $f = r(u)$ *for some* $u \in M^{pi}$*), then there is an element* $u \in \mathcal{G}$ *such that* $l(u) = e$ *and* $r(u) = f$*. Hence, under the same assumption, if* $p \in A^p$ *has the central support projection* $c_M(p) = 1$*, then one can find* $v_k \in \mathcal{G}$ *in such a way that* $\sum_k v_k p v_k^* = 1$*.*

Proof. The latter assertion clearly follows from the former. Since the linear span of $\mathcal{G}$ becomes a dense $*$-subalgebra of M, $e \sim_M f$ implies $eMf \neq \{0\}$ so that there is a $v \in \mathcal{G}$ with $evf \neq 0$. Letting $u_0 := evf$ one has $l(u_0) \leq e$ and $r(u_0) \leq f$, and thus $e - l(u_0) \sim_M f - r(u_0)$ since M is finite. Hence, standard exhaustion argument completes the proof.

To prove the next proposition, Gaboriau's original argument still essentially works purely in operator algebra framework.

Proposition 11. ([14, Proposition I.9; Proposition I.11]) *Suppose that* A *is a MASA in* M*. Then, the following assertions hold true:*

(a) *Let* $\varphi \in M_*$ *be a faithful state with* $\varphi \circ E_A^M = \varphi$*. If a graphing* $\mathcal{U}$ *of* $\mathcal{G}(M \supseteq A)$ *satisfies* $C_\varphi(\mathcal{U}) = C_\varphi(\mathcal{G}(M \supseteq A)) < +\infty$*, then* $\mathcal{U}$ *must be a treeing.*

(b) *If* M *is of finite type I* (*hence* A *is automatically a Cartan subalgebra*) *and* $\tau \in M_*$ *is a faithful tracial state* (*n.b.,* $\tau \circ E_A^M = \tau$ *holds automatically*), *then every treeing* $\mathcal{U}$ *of* $\mathcal{G}(M \supseteq A)$ *satisfies that*

$$C_\tau(\mathcal{U}) = 1 - \tau(e) = C_\tau(\mathcal{G}(M \supseteq A)),$$

where $e \in A^p$ *is arbitrary, maximal, abelian projection of* M (*hence the central support projection* $c_M(e) = 1$).

Proof. (Sketch) (a) Suppose that $\mathcal{U}$ is not a treeing. Then, one can choose a word $v_\ell^{\varepsilon_\ell} \cdots v_1^{\varepsilon_1}$ in $\mathcal{U} \sqcup \mathcal{U}^*$ of reduced form in the formal sense in such a way that $E_A^M(v_\ell^{\varepsilon_\ell} \cdots v_1^{\varepsilon_1}) \neq 0$ but every proper subword $v_i^{\varepsilon_i} \cdots v_j^{\varepsilon_j}$ satisfies that $E_A^M\left(v_i^{\varepsilon_i} \cdots v_j^{\varepsilon_j}\right) = 0$. It is plain to find mutually orthogonal nonzero $e_1, \dots, e_\ell \in A^p$ with $e_k \leq r(v_k^{\varepsilon_k})$ satisfying that $v_k^{\varepsilon_k} e_k v_k^{\varepsilon_k *} = e_{k+1}$ ($k = 1, \dots, \ell - 1$) and $v_\ell^{\varepsilon_\ell} e_\ell v_\ell^{\varepsilon_\ell *} = e_1$, where the following simple fact is needed: If A is a MASA in M, then any $u \in \mathcal{G}(M \supseteq A) \setminus A^{pi}$ has a nonzero $e \in A^p$ such that $e \leq r(v)$ and $e(vev^*) = 0$. Thus, $\mathcal{V} := (\mathcal{U} \setminus \{v_\ell\}) \sqcup \{(l(v_\ell^{\varepsilon_\ell}) - e_1) v_\ell^{\varepsilon_\ell}\}$ becomes a graphing and satisfies $C_\varphi(\mathcal{U}) \gneqq C_\varphi(\mathcal{V})$, a contradiction.

(b) Assume that $M = M_n(\mathbf{C})$. Let $\mathcal{V}$ be a graphing of $\mathcal{G}(M \supseteq A)$. Let $p_1, \dots, p_n \in A^p$ be the mutually orthogonal minimal projections in M, and define the new graphing $\mathcal{V}'$ to be the collection of all nonzero $p_i v p_j$ with $i, j = 1, \dots, n$ and $v \in \mathcal{V}$, each of which is nothing but a standard matrix unit (modulo scalar multiple). Note that $C_\tau(\mathcal{V}) = C_\tau(\mathcal{V}')$ by the construction, and it is plain to see that if $\mathcal{V}$ is a treeing then so is $\mathcal{V}'$ too. We then construct a (non-oriented, geometric) graph whose vertices are $p_1, \dots, p_n$ and whose edges given by $\mathcal{V}'$ with regarding each $p_i v p_j \in \mathcal{V}'$ as an arrow connecting between p_i and p_j. It is plain to see that a sub-collection $\mathcal{U}$ of $\mathcal{V}'$ is a treeing of $\mathcal{G}(M \supseteq A)$ if and only if the subgraph whose edges are given by only $\mathcal{U}$ forms a maximal tree. Therefore, a standard fact in graph theory (see e.g. [28, §§2.3]) tells that $\mathcal{V}'$ contains a treeing $\mathcal{U}$ of $\mathcal{G}(M \supseteq A)$ or $\mathcal{V}'$ becomes a treeing when so is $\mathcal{V}$ itself. Such a treeing is determined as a collection of matrix units $e_{i_1 j_1}, \dots, e_{i_{n-1} j_{n-1}}$ up to scalar multiples with the property that each of $1, \dots, n$ appears at least once in the subindices $i_1, j_1, \dots, i_{n-1}, j_{n-1}$. Hence $C_\varphi(\mathcal{V}) = C_\tau(\mathcal{V}') \geq C_\varphi(\mathcal{U}) = 1 - 1/n$, which implies the desired assertion in the special case of $M = M_n(\mathbf{C})$. The simultaneous central decomposition of $M \supseteq A$ reduces the general case to the above simplest case we have already treated. Proving that any treeing attains $C_\tau(\mathcal{G}(M \supseteq A))$ needs the following simple fact: Let $\mathcal{U}$ be a graphing of $\mathcal{G}(M \supseteq A)$, and set $\mathcal{U}(\omega) := \{u(\omega) : u \in \mathcal{U}\}$ with $u = \int_\Omega^\oplus u(\omega) d\omega$ in the central decomposition of M with $\mathcal{Z}(M) = L^\infty(\Omega) \subseteq A$. Then, $\mathcal{U}$ is $*$-free with respect to E_A^M (or other words, say a treeing) if and only if so is $\mathcal{U}(\omega)$ with respect to $E_{A(\omega)}^{M(\omega)}$ for a.e. $\omega \in \Omega$ with $E_A^M = \int_\Omega^\oplus E_{A(\omega)}^{M(\omega)} d\omega$, see e.g. the proof of [31, Theorem 5.1].

Remark 12. (1) In the above (a), it cannot be avoided to assume that A is a MASA in M, that is, the assertion no longer holds true in the non-principal groupoid case. In fact, let $M := L(\mathbb{Z}_N)$ be the group von Neumann algebra associated with cyclic group $\mathbb{Z}_N$ and $\tau_{\mathbb{Z}_N}$ be the canonical tracial state. Then, $\mathcal{G}(\mathbb{Z}_N) := \mathbb{T}1 \cdot \lambda(\mathbb{Z}_N)$ is a $\tau_{\mathbb{Z}_N}(\,\cdot\,)1$-groupoid, and it is trivial that $C_{\tau_{\mathbb{Z}_N}}(\mathcal{G}(\mathbb{Z}_N)) = C_{\tau_{\mathbb{Z}_N}}(\{\lambda(\bar{1})\})$ with the canonical generator $\bar{1} \in \mathbb{Z}_N$. This clearly provides a counter-example.

(2) Notice that the cost $C_{\tau_G}(\mathcal{G}(G))$ of a group G is clearly the smallest number $n(G)$ of generators of G, and hence Theorem 9 provides a quite natural formula, that is, $n(G \bigstar H) = n(G) + n(H)$. One should here note that the ℓ^2-Betti numbers of discrete groupoids ([15], and also [26]) recover the group ℓ^2-Betti numbers when a given groupoid is a group (see e.g. the approach in [26]).

(3) Assume that M is properly inifinite and A is a Cartan subalgebra in M. Based on the fact that the inclusion $B\big(\ell^2(\mathbb{N})\big) \supseteq \ell^\infty(\mathbb{N})$ can be embedded into $M \supseteq A$, it is not difficult to see that $C_\varphi(\mathcal{G}(M \supseteq A)) = \frac{1}{2}$ for every faithful state $\varphi \in M_*$ with $\varphi \circ E_A^M = \varphi$. Therefore, the idea of costs seems to fit for nothing in the infinite case with general states.

(4) One of the key ingredients in the proof of (b) can be illustrated by

$$M_3(\mathbf{C}) \cong \begin{bmatrix} * & * & \\ * & * & \\ & & * \end{bmatrix} \underset{\begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix}}{\bigstar} \begin{bmatrix} * & & \\ & * & * \\ & * & * \end{bmatrix}$$

which provides the treeing e_{12}, e_{23} of $\mathcal{G}(M \supseteq A)$ with $M = M_3(\mathbf{C})$. This kind of facts are probably known, and specialists in free probability theory are much familiar with similar phenomena in the context of (operator) matrix models of semicircular systems.

Throughout the rest of this section, let us assume that $\mathcal{G}$ is an E_A^M-groupoid with $M = \mathcal{G}''$ and $\tau \in M_*$ is a faithful tracial state with $\tau \circ E_A^M = \tau$. For a given $p \in A^p$ we denote by $p\mathcal{G}p$ the set of pup with $u \in \mathcal{G}$, which becomes an E_{Ap}^{pMp}-groupoid with $E_{Ap}^{pMp} := E_A^M\big|_{pMp}$. The next lemma is technical but quite important, and shown in the same way as in Gaboriau's. It is a graphing counterpart of the well-known construction of induced transformations (see e.g. [13, p.13–14]).

Lemma 13. (cf. [14, Lemme II.8]) *Let $p \in A^p$ be such that the central support projection $c_M(p) = 1$, and $\mathcal{U}$ be a graphing of $\mathcal{G}$. Then, there are a treeing $\mathcal{U}_v$ and a graphing $\mathcal{U}_h$ of $p\mathcal{G}p$ with the following properties*:

(a) *p is an abelian projection of $M_v := A \vee \mathcal{U}_v''$ with $c_{M_v}(p) = 1$*;
(b) *For a graphing $\mathcal{V}$ of $p\mathcal{G}p$, $\mathcal{U}_v \sqcup \mathcal{V}$ becomes a graphing of $\mathcal{G}$*;
(c) *For a graphing $\mathcal{V}$ of $p\mathcal{G}p$, $\mathcal{U}_v \sqcup \mathcal{V}$ is a treeing of $\mathcal{G}$ if and only if so is $\mathcal{V}$*;

(d) $C_\tau(\mathcal{U}) = C_\tau(\mathcal{U}_v) + C_\tau(\mathcal{U}_h)$ *and* $C_\tau(\mathcal{U}_v) = 1 - \tau(p)$.

Proof. (Sketch) Let $\mathcal{U}^{(\ell)}$ be the set of words in $\mathcal{U} \sqcup \mathcal{U}^*$ of reduced form in the formal sense and of length $\ell \geq 1$, and set $q_\ell := \bigvee_{w\in\mathcal{U}^{(\ell)}} wpw^*$. Since A is commutative, we can construct inductively the projections $p_\ell \in A^p$ by $p_\ell := q_\ell(1-p_1-\cdots-p_{\ell-1})$ with $p_0 := p$. Letting $p_0 := p$ we have $\sum_{\ell\geq 0} p_\ell = 1$ thanks to $c_M(p) = 1$. For each $u \in \mathcal{U}$, we define $u_{\ell_1\ell_2} := p_{\ell_1}up_{\ell_2} \in \mathcal{G}$ with $\ell_1, \ell_2 \in \mathbb{N} \sqcup \{0\}$, and consider the new collection $\widetilde{\mathcal{U}} := \bigsqcup_{\ell_1,\ell_2\geq 0} \widetilde{\mathcal{U}}_{\ell_1,\ell_2}$ with $\widetilde{\mathcal{U}}_{\ell_1,\ell_2} := \{u_{\ell_1,\ell_2} : u \in \mathcal{U}\}$ instead of the original $\mathcal{U}$ (without changing the τ-costs). Replacing $u_{\ell_1\ell_2}$ by its adjoint if $\ell_2 \lneqq \ell_1$ we may and do assume that $\widetilde{\mathcal{U}}_{\ell_1,\ell_2} = \emptyset$ as long as when $\ell_2 \lneqq \ell_1$. Then, it is not so hard to see that $p_\ell = \bigvee_{v\in\widetilde{\mathcal{U}}_{\ell-1,\ell}} r(v)$ for every $\ell \geq 1$. Numbering $\widetilde{\mathcal{U}}_{\ell-1,\ell} = \{v_1, v_2, \dots\}$ we construct a partition $p_\ell = \sum_k s_k$ in A^p inductively by $s_k := r(v_k)(1-s_1-\cdots-s_{k-1})$, and set $\widetilde{\mathcal{U}}'_{\ell-1,\ell} := \{v_k s_k\}_k$ and $\widetilde{\mathcal{U}}''_{\ell-1,\ell} := \{v_k(1-s_k)\}_k$. Set $\mathcal{U}_v := \bigsqcup_{\ell\geq 1} \widetilde{\mathcal{U}}'_{\ell-1.\ell}$, and then it is clear that the (right support) projections $r(v)$, $v \in \mathcal{U}_v$, are mutually orthogonal and moreover that $\sum_{v\in\widetilde{\mathcal{U}}'_{\ell-1,\ell}} r(v) = p_\ell$ (hence $\sum_{v\in\mathcal{U}_v} r(v) = 1-p$). Set $\mathcal{U}_v^{[k,\ell]} := \left\{v_{kk+1}\cdots v_{\ell-1\ell} \neq 0 : v_{j-1j} \in \widetilde{\mathcal{U}}'_{j-1,j}\right\}$ with $k \lneqq \ell$, and define $\mathcal{U}_h$ to be the collection of elements in $\mathcal{G}$ of the form, either $v \in \mathcal{U}_{0,0}$ or $w_1vw_2^* \neq 0$ with either $w_1 \in \mathcal{U}_v^{[0,\ell-1]}$, $v \in \widetilde{\mathcal{U}}_{\ell-1,\ell}$, $w_2 \in \mathcal{U}_v^{[0,\ell]}$ ($\ell_1 = \ell_2$ or $\ell_1 \leq \ell_2 - 2$) or $w_1 \in \mathcal{U}_v^{[0,\ell]}$, $v \in \widetilde{\mathcal{U}}''_{\ell-1,\ell}$, $w_2 \in \mathcal{U}_v^{[0,\ell]}$. It is not so hard to verify that all the assertions (a)-(d) hold for the collections $\mathcal{U}_v$, $\mathcal{U}_h$ that we just constructed. (Note here that the trace property of τ is needed only for verifying the assertion (d).)

Remark 14. We should remark that M_v is constructed so that A is a Cartan subalgebra in M_v. Let $\mathcal{G}_v$ be the smallest E_A^M-groupoid that contains $\mathcal{U}_v$, and hence $M_v = \mathcal{G}_v''$ is clear. By the construction of $\mathcal{U}_v$ one easily see that any non-zero word in $\mathcal{U}_v \sqcup \mathcal{U}_v^*$ must be in either $\mathcal{U}_v^{[k,\ell]}$ or its adjoint set so that $p\mathcal{G}_vp = A^{p_i}p$ by Lemma 3. (This pattern of argument is used to confirm that $\mathcal{U}_v$ is a treeing.) Hence, we get $\mathcal{Z}(M_v)p = pM_vp = Ap$, by which with $c_{M_v}(p) = 1$ it immediately follows that $A' \cap M_v = A$, thanks to Lemma 10.

Proposition 15. (cf. [14, Proposition II.6]) *Let $p \in A^p$ be such that the central support projection $c_M(p) = 1$. Then, the following hold true*:

- *$\mathcal{G}$ is treeable if and only if so is $p\mathcal{G}p$*;
- $C_\tau(\mathcal{G}) - 1 = C_{\tau|_{pMp}}(p\mathcal{G}p) - \tau(p)$.

Proof. The first assertion is nothing less than Lemma 13 (c). The second is shown as follows. By Lemma 13 (d), we have $C_\tau(\mathcal{U}) \geq C_\tau(p\mathcal{G}p) + 1 - \tau(p)$ for every graphing $\mathcal{U}$ of $\mathcal{G}$ so that $C_\tau(\mathcal{G}) - 1 \geq C_\tau(p\mathcal{G}p) - \tau(p)$. Let $\varepsilon > 0$ be arbitrary small. Choose a graphing $\mathcal{V}_\varepsilon$ so that $C_\tau(\mathcal{V}_\varepsilon) \leq C_\tau(p\mathcal{G}p) + \varepsilon$. With $\mathcal{U}_v$ as in Lemma 13 the new collection $\mathcal{U}_\varepsilon := \mathcal{U}_v \sqcup \mathcal{V}_\varepsilon$ becomes a graphing of $\mathcal{G}$ by Lemma 13 (b), and hence $C_\tau(\mathcal{G}) \leq C_\tau(\mathcal{U}_\varepsilon) = 1 - \tau(p) + C_\tau(\mathcal{V}_\varepsilon)$ by Lemma 13 (d). Hence, $C_\tau(\mathcal{G}) - 1 \leq C_\tau(\mathcal{V}_\varepsilon) - \tau(p) \leq C_\tau(p\mathcal{G}p) + \varepsilon - \tau(p) \searrow C_\tau(p\mathcal{G}p) - \tau(p)$ as $\varepsilon \searrow 0$.

Corollary 16. ([22, Proposition 1, Theorem 2],[14, Proposition III.3, Lemme III.5]) (a) *Assume that M is of type II_1 and A is a Cartan subalgebra in M. Then, $C_\tau\left(\mathcal{G}(M \supseteq A)\right) \geq 1$, and the equality holds if M is further assumed to be hyperfinite.*

(b) *Assume that M is hyperfinite and A is a Cartan subalgebra in M. Then, every treeing $\mathcal{U}$ of $\mathcal{G}(M \supseteq A)$ (it always exists) satisfies that*

$$C_\tau(\mathcal{U}) = 1 - \tau(e) = C_\tau\left(\mathcal{G}(M \supseteq A)\right),$$

where $e \in A^p$ is arbitrary, maximal, abelian projection of M (hence the central support projection $c_M(e)$ must coincide with that of type I direct summand).

(c) *Let N be a hyperfinite intermediate von Neumann subalgebra between $M \supseteq A$, and assume that A is a Cartan aubalgebra in N. Let $\mathcal{U}$ be a treeing of $\mathcal{G}(N \supseteq A)$ and suppose that $\mathcal{G}$ contains $\mathcal{G}(N \supseteq A)$. Then, for each $\varepsilon > 0$, there is a graphing $\mathcal{U}_\varepsilon$ of $\mathcal{G}$ enlarging $\mathcal{U}$ such that $C_\tau\left(\mathcal{U}_\varepsilon\right) \leq C_\tau(\mathcal{G}) + \varepsilon$.*

Proof. (a) It is known that for each $n \in \mathbb{N}$ there is an $n \times n$ matrix unit system $e_{ij} \in \mathcal{G}(M \supseteq A)$ $(i, j = 1, \dots, n)$ such that all e_{ii}'s are chosen from A^p. Then, Proposition 15 implies that $C_\tau(\mathcal{G}) = C_{\tau|_{e_{11}Me_{11}}}\left(e_{11}\mathcal{G}e_{11}\right) + 1 - \tau\left(e_{11}\right) \geq 1 - \tau\left(e_{11}\right) = 1 - 1/n \nearrow 1$ as $n \to \infty$. The equality in the hyperfinite case clearly follows from celebrated Connes, Feldman and Weiss' theorem [9] (also [23] for its operator algebraic proof).

(b) Choose an incereasing sequence of type I von Neumann subalgebras $A \subseteq M_1 \subseteq \cdots \subseteq M_k \nearrow M$. By Dye's lemma (or Lemma 4), each $u \in \mathcal{U}$ has a unique projection $e_k(u) \in A^p$ such that $e_k(u) \leq l(u)$ and $E^M_{M_k}(u) = e_k(u)u$, where $E^M_{M_k} : M \to M_k$ is the τ-conditional expectation. Set $\mathcal{U}_k := \{e_k(u)u : u \in \mathcal{U}\}$ and $N_k := A \vee \mathcal{U}_k''$ being of type I. Clearly, each $\mathcal{U}_k$ is a treeing of $\mathcal{G}(N_k \supseteq A)$, and hence Proposition 11 (b) says that $C_\tau\left(\mathcal{U}_k\right) = C_\tau\left(\mathcal{G}\left(N_k \supseteq A\right)\right) = 1 - \tau\left(e_k\right)$ for every maximal abelian projection $e_k \in A^p$ of N_k. The non-commutative Martingale convergence theorem (e.g. [8, Lemma 2]) shows that $e_k(u)u = E^M_{M_k}(u) \to u$ in σ-strong* topology, as $k \to \infty$, for every $u \in \mathcal{U}$. Hence, we get $C_\tau\left(\mathcal{U}\right) = \lim_{k\to\infty} C_\tau\left(\mathcal{U}_k\right) = \lim_{k\to\infty} C_\tau\left(\mathcal{G}\left(N_k \supseteq A\right)\right)$ and $N_k = A \vee \mathcal{U}_k'' \nearrow A \vee \mathcal{U}'' = M$. Let $e \in A^p$ be a maximal, abelian projection of M. Then, Proposition 11 (b) and the above (a) show that $C_\tau\left(\mathcal{G}(M \supseteq A)\right) = 1 - \tau(e)$. Since e must be an abelian projection of each N_k, one can choose $e_1, e_2, \dots \in A^p$ in such a way that each e_k is a maximal, abelian projection of N_k and greater than e. It is standard to see that $e = \bigwedge_{k=1}^\infty e_k$ so that $\tau(e_k) \geq \tau\big(\bigwedge_{k'=1}^k e_{k'}\big) \searrow \tau(e)$ as $k \to \infty$. Therefore, $C_\tau(\mathcal{U}) = \lim_{k\to\infty} C_\tau\left(\mathcal{G}\big(N_k \supseteq A\big)\right) = \lim_{k\to\infty}\left(1 - \tau\left(e_k\right)\right) \leq \lim_{k\to\infty}\left(1 - \tau\left(\bigwedge_{k'=1}^k e_{k'}\right)\right) = 1 - \tau(e)$, and then it follows immediately that $C_\tau(\mathcal{U}) = 1 - \tau(e) = C_\tau\left(\mathcal{G}(M \supseteq A)\right)$.

(c) Let $N = N_{\mathrm{I}} \oplus N_{\mathrm{II}_1} \supseteq A = A_{\mathrm{I}} \oplus A_{\mathrm{II}_1}$ be the decomposition into the finite type I and the type II_1 parts. Looking at the decomposition, one can find a projection $p_\varepsilon = p_I \oplus p^\varepsilon_{\mathrm{II}_1} \in A^p$ in such a way that p_{I} is an abelian projection of N_{I} with $c_{N_{\mathrm{I}}}(p_{\mathrm{I}}) = 1_{N_{\mathrm{I}}}$ and $\tau\left(p^\varepsilon_{\mathrm{II}_1}\right) < \varepsilon/2$ with $c_{N_{\mathrm{II}_1}}\left(p^\varepsilon_{\mathrm{II}_1}\right) = 1_{N_{\mathrm{II}_1}}$. Choose a graphing $\mathcal{V}_\varepsilon$ of $p_\varepsilon \mathcal{G} p_\varepsilon$ in such a way that $C_\tau(\mathcal{V}_\varepsilon) \le C_\tau(p_\varepsilon \mathcal{G} p_\varepsilon) + \varepsilon/2$, and then set $\mathcal{U}_\varepsilon := \mathcal{U} \sqcup \mathcal{V}_\varepsilon$. Since $c_N(p_\varepsilon) = 1$, $\mathcal{U}_\varepsilon$ is a graphing $\mathcal{G}$ thanks to Lemma 10. Then, Lemma 13 (b) implies that $C_\tau(\mathcal{V}_\varepsilon) \le C_\tau(p_\varepsilon \mathcal{G} p_\varepsilon) + \varepsilon/2 = C_\tau(\mathcal{G}) - 1 + \tau(p_\varepsilon) + \varepsilon/2 = C_\tau(\mathcal{G}) - (1 - \tau(p_{\mathrm{I}})) + \tau\left(p^\varepsilon_{\mathrm{II}_1}\right) + \varepsilon/2$, and thus by Proposition 11 (b) we get $C_\tau(\mathcal{V}_\varepsilon) \le C_\tau(\mathcal{G}) - C_\tau(\mathcal{U}) + \varepsilon$, which implies the desired assertion.

Remark 17. (1) The proof of (b) in the above also shows "hyperfinite monotonicity," which asserts as follows. Assume that M is hyperfinite and A is a Cartan subalgebra in M. For any intermediate von Neumann subalgebra N between $M \supseteq A$ (in which A becomes automatically a Cartan subalgebra thanks to Dye's lemma, see the discussion above Lemma 5), we have $C_\tau(\mathcal{G}(N \supseteq A)) \le C_\tau(\mathcal{G}(M \supseteq A))$. Furthermore, we have $\lim_{k\to\infty} C_\tau\big(\mathcal{G}(M_k \supseteq A)\big) = C_\tau\big(\mathcal{G}(M \supseteq A)\big)$ for any increasing sequence $A \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_k \nearrow M$ of von Neumann subalgebras. Note that this kind of fact on free entropy dimension was provided by K. Jung [19].

(2) Related to (c) one can show the following (cf. [14, Lemme V.3]): Let $u \in \mathcal{G}$, and $\mathcal{G}_0 \subseteq \mathcal{G}$ be an E^M_A-groupoid, and set $N := \big(r(u)\mathcal{G}_0 r(u)\big)'' \vee \big(u^*\mathcal{G}_0 u\big)''$. If $e \in \big(Ar(u)\big)^p$ has $c_N(e) = r(u)$, then $\mathcal{G}_0 \vee \{u\} = \mathcal{G}_0 \vee \{ue\}$ so that $C_\tau(\mathcal{G}_0 \vee \{u\}) \le C_\tau(\mathcal{G}_0) + \tau(e)$. Here, "$\vee$" means the symbol of generation as E^M_A-groupoid. In fact, by Lemma 10 one finds $v_k \in r(u)\mathcal{G}_0 r(u) \vee u^*\mathcal{G}_0 u$ so that $\sum_k v_k e v_k^* = r(u)$. Since $v_k \in u^*\mathcal{G}_0 u$, one has $v_k = u^* w_k u$ for some $w_k \in l(u)\mathcal{G}_0 l(u)$ so that $\sum_k w_k (ue) v_k^* = u$. This fact can be used in many actual computations, and in fact it tells us that the cost of an E^M_A-groupoid can be estimated by that of its "normal E^M_A-subgroupoid" with a certain condition. Its free entropy dimension counterpart seems an interesting question.

Proposition 18. *Let $\mathcal{G}_1 \supseteq \mathcal{G}_3 \subseteq \mathcal{G}_2$ be E^M_A-groupoids, and let $\mathcal{G} = \mathcal{G}_1 \vee \mathcal{G}_2$ be the smallest E^M_A-groupoid that contains $\mathcal{G}_1$, $\mathcal{G}_2$. If $\mathcal{G}_3''$ is hyperfinite and if A is a MASA in $\mathcal{G}_3''$ (and hence $\mathcal{G}_3 = \mathcal{G}(\mathcal{G}_3'' \supseteq A)$ is automatic), then*

$$C_\tau(\mathcal{G}) \le C_\tau(\mathcal{G}_1) + C_\tau(\mathcal{G}_2) - C_\tau(\mathcal{G}_3).$$

Proof. Choose a treeing $\mathcal{U}$ of $\mathcal{G}_3$ so that $C_\tau(\mathcal{G}_3) = C_\tau(\mathcal{U})$ by Corollary 16 (b). Let $\varepsilon > 0$ be arbitrary small. By Corollary 16 (c), one can choose graphings $\mathcal{U}^{(i)}_\varepsilon$ of $\mathcal{G}_i$ enlarging $\mathcal{U}$, $i = 1, 2$, so that $C_\tau\big(\mathcal{U}^{(i)}_\varepsilon\big) \le C_\tau(\mathcal{G}_i) + \varepsilon/2$. Thus, $C_\tau(\mathcal{G}) \le C_\tau\Big(\big(\mathcal{U}^{(1)}_\varepsilon \setminus \mathcal{U}\big) \sqcup \big(\mathcal{U}^{(2)}_\varepsilon \setminus \mathcal{U}\big) \sqcup \mathcal{U}\Big) = C_\tau\big(\mathcal{U}^{(1)}_\varepsilon\big) + C_\tau\big(\mathcal{U}^{(2)}_\varepsilon\big) - C_\tau(\mathcal{U}) \le C_\tau(\mathcal{G}_1) + C_\tau(\mathcal{G}_2) - C_\tau(\mathcal{G}_3) + \varepsilon \searrow C_\tau(\mathcal{G}_1) + C_\tau(\mathcal{G}_2) - C_\tau(\mathcal{G}_3)$ as $\varepsilon \searrow 0$.

To prove Theorem 9, it suffices to show the inequality $C_\tau(\mathcal{G}) \ge C_\tau(\mathcal{G}_1) + C_\tau(\mathcal{G}_2) - C_\tau(\mathcal{G}_3)$ thanks to Proposition 18. To do so, we begin by providing a simple fact on general amalgamated free products of von Neumann algebras.

Lemma 19. *Let*

$$\big(N, E^N_{N_3}\big) = \big(N_1, E^{N_1}_{N_3}\big) \mathop{\bigstar}_{N_3} \big(N_2, E^{N_2}_{N_3}\big)$$

be an amalgamated free product of (σ-finite) von Neumann algebras, and L_1 and L_2 be von Neumann subalgebras of N_1 and N_2, respectively. Suppose that $\begin{matrix} N_i & \supset & L_i \\ \cup & & \cup \\ N_3 & \supset & N_3 \cap L_i \end{matrix}$ *has faithful normal conditional expectations*

$$E^{N_i}_{L_i} : N_i \to L_i, \quad E^{N_3}_{N_3\cap L_i} : N_3 \to N_3 \cap L_i, \quad E^{L_i}_{N_3\cap L_i} : L_i \to N_3 \cap L_i,$$

and form commuting squares (see e.g. [11, p. 513]) for both $i = 1, 2$. If $L_1 \cap N_3 = L_2 \cap N_3 =: L_3$ and further $N = L_1 \vee L_2$ as von Neumann algebra, then $L_1 = N_2$ and $L_2 = N_2$ must hold true.

Proof. Note that the amalgamated free product

$$\big(L, E^L_{L_3}\big) = \big(L_1, E^L_{L_3}\big) \underset{L_3}{\bigstar} \big(L_2, E^{L_2}_{L_3}\big)$$

can be naturally embedded into $\big(N, E^N_{N_3}\big)$ thanks to the commuting square assumption. Then, it is plain to see that $\begin{matrix} N & \supset & L \\ \cup & & \cup \\ N_i & \supset & L_i \end{matrix}$ form commuting squares too, i.e., $E^N_L\big|_{N_i} = E^{N_i}_{L_i}$, $i = 1, 2$, by which the desired assertion is immediate.

The next technical lemma plays a key rôle in the proof of Theorem 9.

Lemma 20. ([14, IV.37]) *Assume the same setup as in Theorem 9. Let $\mathcal{V} = \mathcal{V}_1 \sqcup \mathcal{V}_2$ be a graphing of $\mathcal{G}$ with collections $\mathcal{V}_1$, $\mathcal{V}_2$ of elements in $\mathcal{G}_1$, $\mathcal{G}_2$, respectively. Then, one can construct two collections $\mathcal{V}'_1$, $\mathcal{V}'_2$ of elements in $\mathcal{G}_3 = \mathcal{G}\,(N_3 \supseteq A)$ in such a way that*

(i) $\mathcal{V}'_1 \sqcup \mathcal{V}'_2$ *is a treeing*;
(ii) $\mathcal{V}_i \sqcup \mathcal{V}'_i$ *is a graphing of $\mathcal{G}_i$ for $i = 1, 2$, respectively.*

Before giving the proof, we illustrate the idea in a typical example. Assume that $M = N_1 \bigstar_{N_3} N_2 \supseteq A$ is of the form: $N_1 := N_1^{(0)} \otimes M_2(\mathbf{C}) \otimes M_2(\mathbf{C})$, $N_2 := N_2^{(0)} \otimes M_2(\mathbf{C}) \otimes M_2(\mathbf{C})$, $N_3 := N_3^{(0)} \otimes M_2(\mathbf{C}) \otimes M_2(\mathbf{C})$ and their common subalgebra $A := A_0 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$, where A_0 is a common Cartan subalgbera of $N_i^{(0)}$, $i = 1, 2, 3$. Denote by $e^{(1)}_{ij} \otimes e^{(2)}_{k\ell}$, $i, j, k, \ell = 1, 2$, the standard matrix units in $M_2(\mathbf{C}) \otimes M_2(\mathbf{C})$. Let $\mathcal{V}_i^{(0)}$ be a collection of elements in $\mathcal{G}_i := \mathcal{G}\big(N_i^{(0)} \supseteq A_0\big)$, $i = 1, 2$, and set

$$\mathcal{V}_1 := \big\{v \otimes 1 \otimes 1 : v \in \mathcal{V}_1^{(0)}\big\} \sqcup \big\{1 \otimes e^{(1)}_{12} \otimes 1\big\},$$
$$\mathcal{V}_2 := \big\{v \otimes 1 \otimes 1 : v \in \mathcal{V}_2^{(0)}\big\} \sqcup \big\{1 \otimes 1 \otimes e^{(1)}_{12}\big\}.$$

Clearly, $\mathcal{V} := \mathcal{V}_1 \sqcup \mathcal{V}_2$ becomes a graphing of $\mathcal{G} := \mathcal{G}_1 \vee \mathcal{G}_2$. In this example, the collections $\mathcal{V}'_1$, $\mathcal{V}'_2$ in the lemma can be chosen for example to be $\big\{1 \otimes e^{(1)}_{11} \otimes e^{(2)}_{12}\big\}$, $\big\{1 \otimes e^{(1)}_{12} \otimes 1\big\}$, respectively. The proof below goes along the line of this procedure with the help of Lemma 19.

Proof. By Lemma 19 with the aid of Lemma 4 (needed to confirm the required commuting square condition, see the explanation after the lemma) the original (ii) is reduced to showing (ii') $N_3 \cap L_1 = N_3 \cap L_2$ with $L_i := A \vee (\mathcal{V}_i \sqcup \mathcal{V}_i')''$, $i = 1, 2$. Choose an increasing sequence of type I von Neumann subalgebras $N_3^{(0)} := A \subseteq N_3^{(1)} \subseteq N_3^{(2)} \subseteq \cdots \subseteq N_3^{(k)} \nearrow N_3$. Let us construct inductively two sequences of collections $\mathcal{V}_1^{(k)}$, $\mathcal{V}_2^{(k)}$ of elements in $\mathcal{G}\big(N_3^{(k)} \supseteq A\big)$ in such a way that

(a) $\mathcal{V}_i^{(k)} \subseteq \mathcal{V}_i^{(k+1)}$ for every k and each $i = 1, 2$;
(b) $\mathcal{V}_1^{(k)} \sqcup \mathcal{V}_2^{(k)}$ is a treeing for every k;
(c) letting $L_i^{(k)} := A \vee \left(\mathcal{V}_i \sqcup \mathcal{V}_i^{(k)}\right)''$, $i = 1, 2$, we have
  (1) $\mathcal{V}_1^{(k)} \sqcup \mathcal{V}_2^{(k)} \subseteq L_1^{(k)} \cap L_2^{(k)}$ for every k,
  (2) $N_3^{(k)} \cap L_1^{(k)} \subseteq L_2^{(k)}$ for every even k;
  (3) $N_3^{(k)} \cap L_2^{(k)} \subseteq L_1^{(k)}$ for every odd k.

((c-1) is needed only for the inductive precedure.) If such collections were constructed, then $\mathcal{V}_i' := \bigcup_k \mathcal{V}_i^{(k)}$, $i = 1, 2$, would be desired ones. In fact, any word in $\big(\mathcal{V}_1' \sqcup \mathcal{V}_2'\big) \sqcup \big(\mathcal{V}_1' \sqcup \mathcal{V}_2'\big)^*$ of reduced form in the formal sense is in turn one in $\big(\mathcal{V}_1^{(k)} \sqcup \mathcal{V}_2^{(k)}\big) \sqcup \big(\mathcal{V}_1^{(k)} \sqcup \mathcal{V}_2^{(k)}\big)^*$ for some finite k thanks to (a), and thus (i) follows from (b). For each pair k_1, k_2, the above (c-2), (c-3) imply, with $k_1, k_2 \leq 2k$, that

$$N_3^{(k_1)} \cap L_1^{(k_2)} \subseteq N_3^{(2k)} \cap L_1^{(2k)} \subseteq N_3^{(2k)} \cap L_2^{(2k)} \subseteq N_3 \cap L_2;$$
$$N_3^{(k_1)} \cap L_2^{(k_2)} \subseteq N_3^{(2k+1)} \cap L_2^{(2k+1)} \subseteq N_3^{(2k+1)} \cap L_1^{(2k+1)} \subseteq N_3 \cap L_1.$$

Hence,

$$\overline{\bigcup_{k_1,k_2} N_3^{(k_1)} \cap L_1^{(k_2)}}^{\sigma\text{-s}} \subseteq N_3 \cap L_2, \quad \overline{\bigcup_{k_1,k_2} N_3^{(k_1)} \cap L_2^{(k_2)}}^{\sigma\text{-s}} \subseteq N_3 \cap L_1.$$

Note here that $N_3 \cap L_i = \overline{\bigcup_{k_1,k_2} N_3^{(k_1)} \cap L_i^{(k_2)}}^{\sigma\text{-s}}$ for both $i = 1, 2$, since all

$$\begin{array}{ccc} N_i & \supset & L_i \\ \cup & & \cup \\ N_3 & \supset & N_3 \cap L_i, \end{array} \quad \begin{array}{ccc} N_i & \supset & L_i \\ \cup & & \cup \\ N_3^{(k)} & \supset & N_3^{(k)} \cap L_i, \end{array} \quad \begin{array}{ccc} N_i & \supset & L_i^{(k_2)} \\ \cup & & \cup \\ N_3^{(k_1)} & \supset & N_3^{(k_1)} \cap L_i^{(k_2)} \end{array}$$

form commuting squares for every k, k_1, k_2 and each $i = 1, 2$, thanks to Dye's lemma (or Lemma 4); note here that A is assumed to be a Cartan subalgebra in N_3. Then, (ii') follows immediately.

Set $\mathcal{V}_1^{(0)} = \mathcal{V}_2^{(0)} := \emptyset$. Assume that we have already constructed $\mathcal{V}_1^{(j)}$, $\mathcal{V}_2^{(j)}$, $j = 1, 2, \ldots, k$, and that the next $k+1$ is even (the odd case is also done in the same way). Consider

$$K_1 := N_3^{(k+1)} \cap L_1^{(k)} \supseteq K_0 := N_3^{(k+1)} \cap L_1^{(k)} \cap L_2^{(k)} \, (\supseteq A),$$

which are clearly of finite type I. Then, one can choose an abelian projection $p \in A^p$ of K_0 with the central support projection $c_{K_0}(p) = 1$. Also, one can find a treeing $\mathcal{U}_p$ of $\mathcal{G}$ $(pK_1p \supseteq Ap)$, see the proof of Proposition 11 (b). Set $\mathcal{V}_1^{(k+1)} := \mathcal{V}_1^{(k)}$, $\mathcal{V}_2^{(k+1)} := \mathcal{V}_2^{(k)} \sqcup \mathcal{U}_p$, which are desired ones in the $k+1$ step. In fact, (a) is trivial, and (b) follows from the (rather trivial) fact that K_0 and pK_1p are $*$-free with respect to E_A^M and (c-1) for k. Note that $N_3^{(k+1)} \cap L_1^{(k+1)} = N_3^{(k+1)} \cap L_1^{(k)} = K_1 = K_0 \vee pK_1p \subseteq L_2^{(k)} \vee \mathcal{U}_p'' = L_2^{(k+1)}$ (by Lemma 10 it follows from $c_{K_0}(p) = 1$ that $K_1 = K_0 \vee pK_1p$), which is nothing but (c-2). Finally, (c-1) follows from the assumption of induction together with that $\mathcal{U}_p \subseteq pK_1p \subseteq L_1^{(k)}$.

One of the important ideas in Gaboriau's argument is the use of "adapted systems." It is roughly translated to amplification/reduction procedure in operator algebra framework.

Proof. [Proof of Theorem 9] (Step I: Approximation) By Proposition 18, it suffices to show that $C_\tau(\mathcal{G}) \geq C_\tau(\mathcal{G}_1) + C_\tau(\mathcal{G}_2) - C_\tau(\mathcal{G}_3)$ modulo "arbitrary small error." Let $\varepsilon > 0$ be arbitrary small. There is a graphing $\mathcal{V}$ of $\mathcal{G}$ with $C_\tau(\mathcal{V}) \leq C_\tau(\mathcal{G}) + \varepsilon/3$, and we choose a graphing $\mathcal{U}_i$ of $\mathcal{G}_i$ with $C_\tau(\mathcal{U}_i) < +\infty$ (thanks to $C_\tau(\mathcal{G}_i) < +\infty$) for each $i = 1, 2$, and set $\mathcal{U} := \mathcal{U}_1 \sqcup \mathcal{U}_2$. Since $C_\tau(\mathcal{U}) = \sum_{u\in\mathcal{U}} \tau(l(u)) < +\infty$, there is a finite sub-collection $\mathcal{U}_0$ of $\mathcal{U}$ with $\sum_{u\in\mathcal{U}\setminus\mathcal{U}_0} \tau(l(u)) \leq \varepsilon/3$. Since both $\mathcal{V}$ and $\mathcal{U}$ are graphings of $\mathcal{G}$, we may and do assume, by cutting each $v \in \mathcal{V}$ by suitable projections in A^p based on Lemma 3, that each $v \in \mathcal{V}$ has a word $w(v)$ in $\mathcal{U} \sqcup \mathcal{U}^*$ of reduced form in the formal sense and a $a(v) \in A^{pi}$ with $v = a(v)w(v)$. Denote by $w_0 := 1, w_1, w_2, \dots$ the all words in $\mathcal{V} \sqcup \mathcal{V}^*$ of reduced form, and by Lemma 3 again, each $u \in \mathcal{U}$ is described as $u = \sum_j p_k(u)a_k(u)w_k$ in σ-strong topology, where $l(u) = \sum_k p_k(u)$ in A^p, the $a_k(u)$'s are in A^{pi}, and $p_k(u)u = p_k(u)a_k(u)w_k$ for every k. Then, we can choose a $k_0 \in \mathbb{N}$ (depending only on the finite collection $\mathcal{U}_0$) in such a way that $\sum_{k\geq k_0+1} \tau(p_k(u)) \leq \varepsilon/(3\sharp(\mathcal{U}_0))$ for all $u \in \mathcal{U}_0$. Set

$$\mathcal{X} := \{v \in \mathcal{V} : v \text{ appears in } w_1, \dots, w_{k_0}\};$$
$$\mathcal{Y} := \{p(u)u : u \in \mathcal{U}_0\} \sqcup (\mathcal{U} \setminus \mathcal{U}_0)$$

with $p(u) := \sum_{k\geq k_0+1} p_k(u)$ for $u \in \mathcal{U}_0$. Since $u = \sum_{j=0}^{k_0} p_k(u)a_k(u)w_k + p(u)u$ for all $u \in \mathcal{U}_0$, the collection $\mathcal{Z} := \mathcal{X} \sqcup \mathcal{Y}$ becomes a graphing of $\mathcal{G}$. We have

$$C_\tau(\mathcal{Z}) = C_\tau(\mathcal{X}) + C_\tau(\mathcal{Y}) \leq C_\tau(\mathcal{V}) + C_\tau(\mathcal{Y}) \leq C_\tau(\mathcal{G}) + \varepsilon. \tag{5}$$

Clearly, $\mathcal{Y}$ is decomposed into two collections $\mathcal{Y}_1$, $\mathcal{Y}_2$ of elements in $\mathcal{G}_1$, $\mathcal{G}_2$, respectively, as $\mathcal{Y} = \mathcal{Y}_1 \sqcup \mathcal{Y}_2$, while $\mathcal{X}$ not in general. Thus, we replace $\mathcal{X}$ by a new "decomposable" graphing in a sufficiently large amplification of $M \supseteq A$ for the use of Lemma 20.

(Step II: Adapted system/Amplification) Notice that each $v \in \mathcal{X}\big(\subseteq \mathcal{V}\big)$ is described as

$$v = a(v)w(v) = a(v)u_{n(v)}(v)^{\delta_{n(v)}(v)} \cdots u_1(v)^{\delta_1(v)},$$

where $n(v) \in \mathbb{N}$ and $u_i(v) \in \mathcal{U}$, $\delta_i(v) \in \{1, *\}$ $(i = 1, \dots, n(v))$. Cutting each $u_i(v)$ by a suitable projection in A^p and replacing $u_i(v)$ by $u_i(v)^*$ if $\delta_i(v) = *$, etc., we may and do assume the following: $r\left(u_{i+1}(v)\right) = l\left(u_i(v)\right)$ $(i = 1, \dots, n(v) - 1)$; $l(v) = l\left(a(v)\right) (= r\left(a(v)\right)) = l\left(u_{n(v)}(v)\right)$ and $r(v) = r\left(u_1(v)\right)$; and each $u_i(v)$ is of the form either eu or eu^* with $e \in A^p$, $u \in \mathcal{U}$. In what follows, we "reveal" all words $u_{n(v)}(v) \cdots u_1(v)$'s as follows. Set $n := 1 + \sum_{v \in \mathcal{X}} \left(n(v) - 1\right) < +\infty$, and choose a partition $\{2, \dots, n\} = \bigsqcup_{v \in \mathcal{X}} I_v$ $\sharp(I_v) = n(v) - 1$. Denote by e_{ij} the standard matrix units in $M_n(\mathbf{C})$, by Tr_n the canonical non-normalized trace on $M_n(\mathbf{C})$, and by $E_{\mathbf{C}^n}^{M_n(\mathbf{C})}$ the Tr_n-conditional expectation from $M_n(\mathbf{C})$ onto the diagonal matrices $\mathbf{C}^n \subseteq M_n(\mathbf{C})$. Let $M^n := M \otimes M_n(\mathbf{C}) \supseteq A^n := A \otimes \mathbf{C}^n$ be the n-amplifications and write $\tau^n := \tau \otimes \mathrm{Tr}_n \in M_*^n$. For each $v \in \mathcal{X}$, we define the $n(v)$ elements $\tilde{u}_1(v), \dots, \tilde{u}_{n(v)}(v) \in \mathcal{G}\left(M^n \supseteq A^n\right)$ by

$$\begin{aligned}
\tilde{u}_1(v) &:= u_1(v) \otimes e_{i_2 1}, \\
\tilde{u}_2(v) &:= u_2(v) \otimes e_{i_3 i_2}, \\
&\vdots \\
\tilde{u}_{n(v)}(v) &:= u_{n(v)}(v) \otimes e_{1 i_{n(v)}}
\end{aligned}$$

with $I_v = \left\{i_2, \dots i_{n(v)}\right\}$. Set $\widetilde{\mathcal{Z}} := \widetilde{\mathcal{X}} \sqcup \{y \otimes e_{11} : y \in \mathcal{Y}\}$ as a collection of elements of $\mathcal{G}\left(M^n \supseteq A^n\right)$ with $\widetilde{\mathcal{X}} := \{\tilde{u}_i(v) : i = 1, \dots, n(v), v \in \mathcal{X}\}$, and

$$P := 1 \otimes e_{11} + \sum_{v \in \mathcal{X}} \sum_{i=1}^{n(v)-1} l\left(\tilde{u}_i(v)\right) = 1 \otimes e_{11} + \sum_{v \in \mathcal{X}} \sum_{i=2}^{n(v)} r\left(\tilde{u}_i(v)\right).$$

By straightforward calculation we have

$$C_\tau\left(\mathcal{Z}\right) - 1 = C_{\tau^n}\left(\widetilde{\mathcal{Z}}\right) - \tau^n(P). \tag{6}$$

(Step III: Reduction) Set $\widetilde{A} := A^n P$ and $\widetilde{M} := P M^n P$. Clearly, $\widetilde{M}$ is generated by the $g \otimes e_{11}$'s with $g \in \mathcal{G}$ and the $l\left(u_k(v)\right) \otimes e_{i_{k+1} 1}$'s with $k = 1, \dots, n(v) - 1$, $v \in \mathcal{V}$. Set $\tilde{\tau} := \tau^n\big|_{\widetilde{M}}$, and the $\tilde{\tau}$-conditional expectation $E_{\widetilde{A}}^{\widetilde{M}} : \widetilde{M} \to \widetilde{A}$ is given by the restriction of $E_A^M \otimes E_{\mathbf{C}^n}^{M_n(\mathbf{C})}$ to $\widetilde{M}$. Let $\widetilde{\mathcal{G}}$ be the smallest $E_{\widetilde{A}}^{\widetilde{M}}$-groupoid that contains $\left\{g \otimes e_{11} : g \in \mathcal{G}\right\} \sqcup \left\{l\left(u_k(v)\right) \otimes e_{i_{k+1} 1} : k = 1, \dots, n(v) - 1, v \in \mathcal{V}\right\}$. Also, for each $i = 1, 2, 3$, let $\widetilde{\mathcal{G}}_i$ be the smallest $E_{\widetilde{A}}^{\widetilde{M}}$-groupoid that contains $\left\{g \otimes e_{11} : g \in \mathcal{G}_i\right\} \sqcup \left\{l\left(u_k(v)\right) \otimes e_{i_{k+1} 1} : k = 1, \dots, n(v) - 1, v \in \mathcal{V}\right\}$ and set $\widetilde{N}_i := \widetilde{\mathcal{G}}_i''$. Then, it is clear that $\widetilde{N}_i = P\left(N_i \otimes M_n(\mathbf{C})\right) P = \left(N_i \otimes M_n(\mathbf{C})\right) \cap \widetilde{M}$, $i = 1, 2, 3$. In particular, $\widetilde{A}$ is a Cartan subalgebra in $\widetilde{N}_3$, and thus $\widetilde{\mathcal{G}}_3 = \mathcal{G}\left(\widetilde{N}_3 \supseteq \widetilde{A}\right)$. We have $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}_1 \vee \widetilde{\mathcal{G}}_2$, i.e., the smallest $E_{\widetilde{A}}^{\widetilde{M}}$-groupoid that contains both $\widetilde{\mathcal{G}}_1$ and $\widetilde{\mathcal{G}}_2$. Here, simple facts are in order.

(a) $N_1 \otimes M_n(\mathbf{C})$ and $N_2 \otimes M_n(\mathbf{C})$ are free with amalgamation over $N_3 \otimes M_n(\mathbf{C})$ inside $M \otimes M_n(\mathbf{C})$ with subject to $E^M_{N_3} \otimes \mathrm{id}_{M_n(\mathbf{C})}$;
(b) $v \otimes e_{11} = (a(v) \otimes e_{11}) \cdot \widetilde{u}_{n(v)}(v) \cdots \widetilde{u}_1(v)$ for every $v \in \mathcal{X}$;
(c) $l\,(u_k(v)) \otimes e_{i_{k+1}1} = (\widetilde{u}_k(v) \cdots \widetilde{u}_1(v))\,((u_k(v) \cdots u_1(v)) \otimes e_{11})$ for every $k = 1, \dots, n(v) - 1$ and $v \in \mathcal{X}$.

By (a), we have $\widetilde{M} \cong \widetilde{N}_1 \bigstar_{\widetilde{N}_3} \widetilde{N}_2$ with respect to the restriction of $E^M_{N_3} \otimes \mathrm{id}_{M_n(\mathbf{C})}$ to $\widetilde{M}$ (giving the $\tilde{\tau}$-conditional expectation onto $\widetilde{N}_3$). By (b) and (c), it is plain to see that $\widetilde{\mathcal{Z}}$ is a graphing of $\widetilde{\mathcal{G}}$. Moreover, by its construction, it is decomposable, that is, $\widetilde{\mathcal{Z}} = \widetilde{\mathcal{Z}}_1 \sqcup \widetilde{\mathcal{Z}}_2$ with collections $\widetilde{\mathcal{Z}}_1$, $\widetilde{\mathcal{Z}}_2$ of elements in $\widetilde{\mathcal{G}}_1$, $\widetilde{\mathcal{G}}_2$, respectively. Hence, Lemma 20 shows that there is a treeing $\widetilde{\mathcal{Z}}' = \widetilde{\mathcal{Z}}'_1 \sqcup \widetilde{\mathcal{Z}}'_2$ in $\widetilde{\mathcal{G}}_3 = \mathcal{G}(\widetilde{N}_3 \supseteq \widetilde{A})$ with the property that $\widetilde{\mathcal{Z}}_i \sqcup \widetilde{\mathcal{Z}}'_i$ is a graphing of $\widetilde{\mathcal{G}}_i$ for each $i = 1, 2$. Therefore, by Corollary 16 (b) (or Remark 17), we have

$$C_{\tilde{\tau}}(\widetilde{\mathcal{Z}}) = C_{\tilde{\tau}}(\widetilde{\mathcal{Z}}_1 \sqcup \widetilde{\mathcal{Z}}'_1) + C_{\tilde{\tau}}(\widetilde{\mathcal{Z}}_2 \sqcup \widetilde{\mathcal{Z}}'_2) - C_{\tilde{\tau}}(\widetilde{\mathcal{Z}}') \geq C_{\tilde{\tau}}(\widetilde{\mathcal{G}}_1) + C_{\tilde{\tau}}(\widetilde{\mathcal{G}}_2) - C_{\tilde{\tau}}(\widetilde{\mathcal{G}}_3).$$

It is trivial that $c_{\widetilde{N}_i}(1 \otimes e_{11}) = 1_{\widetilde{M}}$ for all $i = 1, 2, 3$ with $1_{\widetilde{M}} = P$, and that $\{g \otimes e_{11} : g \in \mathcal{G}_i\} = (1 \otimes e_{11})\,\widetilde{\mathcal{G}}_i\,(1 \otimes e_{11})$ for every $i = 1, 2, 3$. Hence, (5),(6) and Propsotion 15 altogether imply that

$$\begin{aligned} C_\tau(\mathcal{G}) + \varepsilon &\geq C_\tau(\mathcal{Z}) \\ &= C_{\tilde{\tau}}(\widetilde{\mathcal{Z}}) - \tilde{\tau}(1_{\widetilde{M}}) + 1 \\ &\geq C_{\tilde{\tau}}(\widetilde{\mathcal{G}}_1) + C_{\tilde{\tau}}(\widetilde{\mathcal{G}}_2) - C_{\tilde{\tau}}(\widetilde{\mathcal{G}}_3) - \tilde{\tau}(1_{\widetilde{M}}) + 1 \\ &= C_\tau(\mathcal{G}_1) + C_\tau(\mathcal{G}_2) - C_\tau(\mathcal{G}_3). \end{aligned}$$

References

1. S.R. Adams, *Trees and amenable equivalence relations*, Ergod. Th. Dynam. Sys., **10** (1990), 1–14.
2. S.R. Adams and R.J. Spatzier, *Kazhdan groups, cocycles and trees*, Amer. J. Math. **112** (1990), 271–287.
3. H. Aoi, *A construction of equivalence subrelations for intermediate subalgebras*, J. Math. Soc. Japan **55** (2003), 713–725.
4. F. Boca, *On the method of constructing irreducible finite index subfactors of Popa*, Pacific J. Math. **161** (1993), 201–231.
5. E. Blanchard and K.J. Dykema, *Embeddings of reduced free products of operator algebras*, Pacific J. Math. 199 (2001), 1-19.
6. M. Choda, *On the conditional expectation of a partial isometry in a certain von Neumann algebra*, Proc. Japan Acad. **41** (1965) 277–279.
7. M. Choda, *Group factors of the Haagerup type*, Proc. Japan Acad. Ser. A Math. Sci., **59** (1983), 174–177.
8. A. Connes, *On hyperfinite factors of type* III_0 *and Krieger's factors*, J. Funct. Anal., **18** (1975), 318–327.

9. A. Connes, J. Feldman and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergod. Th. & Dynam. Sys., **1** (1981), 431–450.
10. H. Dye, *On groups of measure preserving transformations, II*, Amer. J. Math. **85** (1963) 551–576.
11. D. Evans and Y. Kawahigashi, *Quantum Symmetries on Operator Algebras*, Oxford Mathematical Monographs, Oxford University Press, New York, 1998.
12. J. Feldman and C.C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras*, Trans. Amer. Math. Soc. **234** (1977), Part I, 289–324; Part II, 325–359.
13. N.A. Friedman, *Introduction to Ergodic Theory*, Van Nostrand Reinhold Mathematical Studies, No. 29. Van Nostrand Reinhold Co., New York-Toronto, Ont.-London, 1970.
14. D. Gaboriau, *Coût des relations d'équivalence et des groupes*, Invent. Math. **139** (2000), 41–98.
15. D. Gaboriau, *Invariants l^2 de relations d'équivalence et de groupes*, Publ. Math. Inst. Hautes Études Sci. No. 95 (2002), 93–150.
16. P. Hahn, *Haar measure for measure groupoids*; *The regular representations of measure groupoids*, Trans. Amer. Math. Soc., **242** (1978), 1–33, 35–72.
17. U. Haagerup, *An example of a non nuclear C^*-algebra, which has the metric approximation property*, Invent. Math., **50** (1979), 279–293.
18. P. Jolissaint, *Haagerup approximation property for finite von Neumann algebras*, J. Operator Theory, **48** (2002), 549–571.
19. K. Jung, *The free entropy dimension of hyperfinite von Neumann algebras*, Trans. Amer. Math. Soc., **355** (2003), 5053–5089.
20. A. Kechris and B. Miller, *Lectures on Orbit Equivalence*, Lecture Notes in Math., **1852**, Springer-Verlag, Berlin, 2004.
21. H. Kosaki, *Free products of measured equivalence relations*, J. Funct. Anal., **207** (2004), 264–299.
22. G. Levitt, *On the cost of generating an equivalence relation*, Ergodic Theory Dynam. Systems **15** (1995), 1173–1181.
23. S. Popa, *Notes on Cartan subalgebras in type* II_1 *factors*, Math. Scand., **57** (1985), 171–188.
24. S. Popa, *On a class of type II_1 factors with Betti numbers invariants*, math.OA/0209130, to appear in Ann. Math.
25. J. Renault, *A Groupoid Approach to C^*-algebras*, Lecture Notes in Mathematics, **793**, Springer, Berlin, 1980.
26. R. Sauer, *L^2-Betti numbers of discrete measurable groupoids*, math.AT/0312411.
27. J.-L. Sauvageot, *Sur le produit tensoriel relatif d'espaces de Hilbert*, J. Operator Theory, **9** (1983), 237–252.
28. J-P. Serre, *Trees*, Springer-Verlag, Berlin-New York, 1980.
29. D. Shlyakhtenko, *Free Fisher information with respect to a completely positive map and costs of equivalence relations*, Commun. Math. Phys., **218** (2001), 133–152.
30. D. Shlyakhtenko, *Microstates free entropy and costs of equivalence relations*, Duke Math. J., **118** (2003), 375–425.
31. Y. Ueda, *Amalgamated free product over Cartan subalgebra, II. Supplementary results & examples*, Operator algebras and applications, 239–265, in Adv. Stud. Pure Math., **38**, Math. Soc. Japan, Tokyo, 2004.
32. D.V. Voiculescu, K.J. Dykema and A. Nica, *Free Random Variables*, CRM Monograph Series, **1**. American Mathematical Society, Providence, RI, 1992.